V. THE SOLDIERS' GAME

187. Rules of the game. The games of §§170 and 181 are simplifications of the so-called soldiers' game, which was published in a French military magazine in 1886, when it attracted much attention because its simplicity of design is coupled with an extraordinary variety of possibilities which make it a substantially more difficult game than the games of dwarfs of §§170 and 181.

The soldiers' game is played on a board with eleven circles, as shown in *Figure 91*. Three white pieces are placed on circles 1, 2, and 4, after which a black piece is allowed a choice of one of the eight remaining circles. White wins if he manages to encircle Black on 11; otherwise White loses. For the rest, the rules are the same as for the games of §§170 and 181.

The increase in the number of circles makes it an extremely difficult matter to arrive at a comprehensive view of the game without making a preliminary analysis. This will establish that White can win, no matter on what circle Black chooses to open the game.

However, this can indeed become an extraordinarily difficult task for White if Black makes several retreats to the circle 11. White then need not be concerned to prevent a breakthrough by Black, and if this is all he considers, White has a choice of several moves; there are some positions where only one of these moves will allow him to retain the opposition.

Because this strategy forces White to make a larger number of moves, there is a fair chance that he will lose the opposition and there will be a repetition of moves, if Black plays correctly, all the more so since in many cases the moves that will allow White to retain the opposition are far from obvious ones.

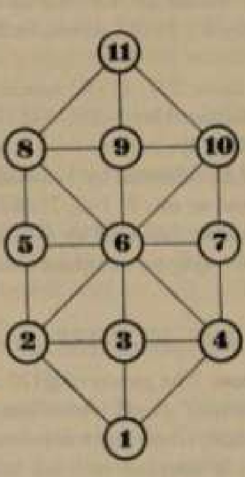

Fig. 91

188. Winning positions. To be sure of following correct lines of play, White must be aware of a number of winning positions (positions in which White will win, or more accurately can win, if it is not his move), since these are the positions that White should aim at producing.

These positions, which are found by the same principle as before (starting out with the final situation) are shown in *Figure 92* (where winning positions that White does not need are omitted); Black's circle has been indicated by a number which also serves to give the number of moves in which White can win, while the double circles indicate the positions of the white pieces.

From the initial position, White can attain one of the winning positions of diagram 17, unless Black opens on circle 6. If Black begins with 6, White can secure one of the two winning positions which are indicated in diagram 18, and this is true also when Black

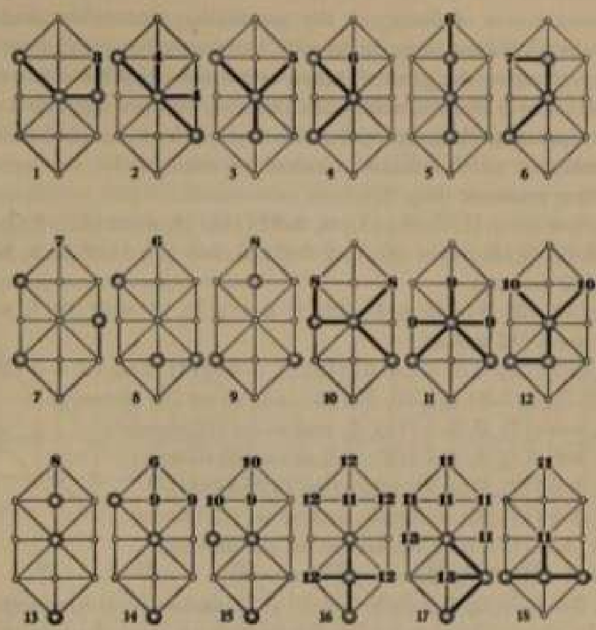

Fig. 92

opens with 11. This makes it clear that White can indeed win against every opening move by Black, in at most 14 moves (14 moves if Black opens on 3, and 12 moves otherwise).

Among the winning positions there are some of the critical type, in which the player who has the move will be the loser. These positions have again been indicated by using heavy lines to connect the circles for the white pieces and the circle for the black piece.

189. Course of the game. To play a successful game, all that White has to do is to keep securing one of the winning positions of Figure 92. The best course for Black is to make his moves in the way which keeps making White need the largest possible number of moves for a win, since this gives White a larger number of opportunities to make a mistake, and likewise involves him in a need to make several transverse moves (for instance, from 2 to 3 or from 8 to 9), something which he may overlook in pressing onwards to his goal.

We shall now set out the sequences of moves for the six possible opening moves of Black, on the assumption that White and Black will always choose their moves with the respective aims of curtailing and prolonging the game as much as possible. If White has only one correct move, and this a very obvious one, we have signalized this by attaching an exclamation point. Each move by White is followed by parentheses which enclose a reference number for the associated winning positions (Fig. 92).

8, 4–6 (17), 11, 2–5! (15), 9, 5–8! (14), 11, 8–9! (13), 8, 1–2 (6), 11, 2–3! (5), 8, 9–10! (3), 9, 3–2 (2), 8, 2–5 (1), 11, 6–8, 9, 5–6, 11, 6–9 (12 moves);

6, 1–3 (18), 8 or 10, 4–6! (12), 9, 3–4! (11), 11, 6–9! (9), 8, 4–6 (6), and so on (12 moves);

6, 1–3 (18), 8 or 10, 4–6! (12), 9, 3–4! (11), 8 or 10, 4–7 (10), 11, 6–8! (7), 9, 7–6! (4), 10, 2–3 (3), and so on (12 moves);

5, 4–6 (17), 8, 2–5 (15), 9, and so on (12 moves);

5, 4–6 (17), 8, 1–3 (12), and so on (12 moves);

9, 4–6 (17), 8 or 11, and so on (12 moves);

9, 4–6 (17), 10, 1–3 (12), and so on (12 moves);

11, 4–6 (17), 8 or 10, and so on (12 moves);

11, 1–3 (18), 8, and so on (12 moves);

3, 2–6 (17), 2, 4–3 (16), 5, 3–2! (17), and so on (14 moves).

The phrase "and so on" indicates that an equivalent position has been discussed earlier in the list.

Black can prolong the game the most by beginning on circle 3. Yet this is not a way for him to make things more difficult for White, because it virtually forces White's first two moves. Black can also play in a way which allows White to win in a smaller number of moves, which require White to set up the winning position 8 (which has not yet appeared in our sequences of moves), as in the following example: 6, 1–3 (18), 8 or 10, 4–6 (12), 11, 6–10! (8), 8, 2–6 (3), and so on (9 moves).

The previous results will show us how easy it can be for White to lose the opposition. When Black retreats repeatedly to circle 11, the game becomes extremely difficult for White, and something which is beyond the mental analysis even of a great chess master.

190. Other winning positions. Apart from the winning positions shown in Figure 92, there are other winning positions for White (when it is not his move), as indicated in *Figure 93*. By making use of these further positions (which in several cases increase the required number of moves), White can make it still more difficult for Black to discover the correct way of playing.

The last diagram shows that White could reply with 2–3 (or 4–3) to each of Black's opening moves (except 3); but this would not break any new ground. Furthermore, in the position 3, 6, 9, 8 (with Black on 8) White can reply with 3–2, which does not get White anywhere forward, either, because this is followed by 11, 2–3, 8. So there should be a stipulation that White becomes the loser when the same position has occurred three times, say. White has to play still more carefully if there is a stipulation that he becomes the loser when the same position has arisen twice over.

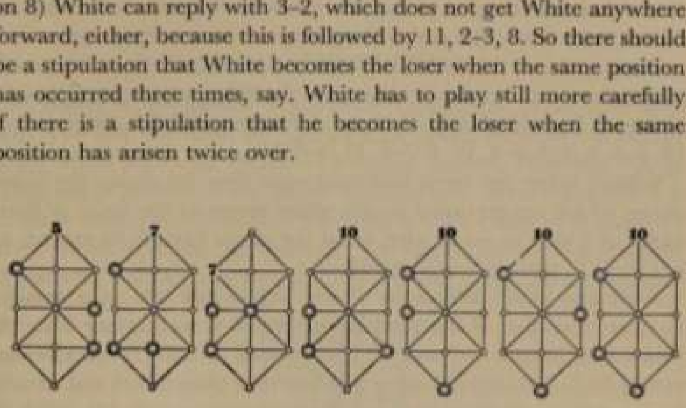

Fig. 93

191. Modified soldiers' game. We modify the soldiers' game of §187 by requiring White to move every one of his three pieces, in any order he likes, whenever it is his turn to play. White's goal is to encircle Black on circle 11, which requires the White pieces to be on the circles 8, 9, 10. Even in his last play, when Black is on 11, White has to move every one of his three pieces in order to win; if White reaches the final position by moving one or two of his pieces only, this makes him the loser. Initially the white pieces are placed on circles 1, 2, and 4, and the black piece on any other circle, after which it is White's move. Now the question is: What initial positions of Black's piece will allow White a win, and how will White obtain this?

-21-

Hare and Hounds

I like the hunting of the hare
Better than that of the fox.

Wilfred Scawen Blunt, *The Old Squire.*

The French Military Hunt

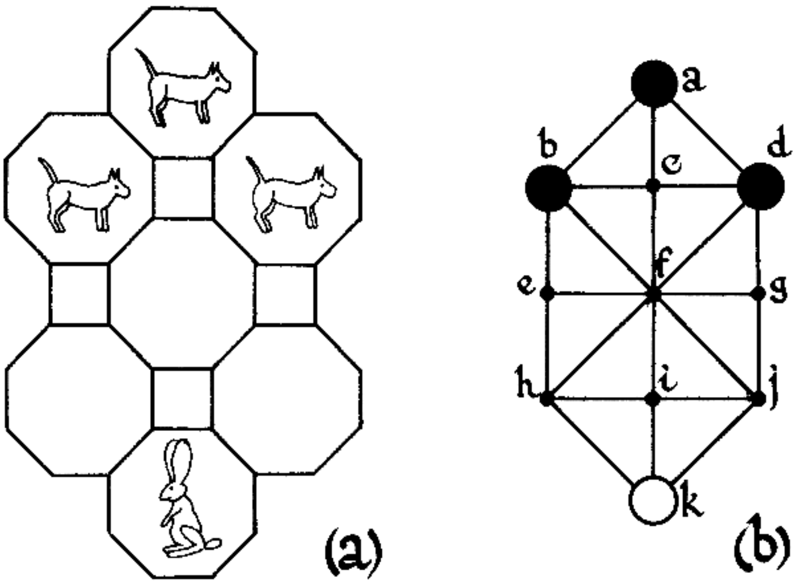

Figure 1. The French Military Hunt. Hare and Hounds on the Small Board.

This little game is very like Fox and Geese. It features a hunter whose three houn*d*s (*d*ogs) try to trap a ha*r*e (*r*abbit) on the board shown in Fig. 1(a). If you can't persuade enough animals to make the right manoeuvres, you can play with four coins on the nodes of the equivalent board shown in Fig. 1(b). It becomes more interesting on the larger board of Fig. 2. At each turn the hunter moves any one hound to a neighboring empty place, and the hare makes a similar move. However the hounds, starting from the top, may not retreat, although a hound may go back and forth horizontally as between e and f in Fig. 1(b). The hare is completely free to advance or retreat or move horizontally. The hounds win by trapping the hare so that he cannot move at his turn. If the hounds fail to advance in ten consecutive moves, the game is usually declared a win for the hare.

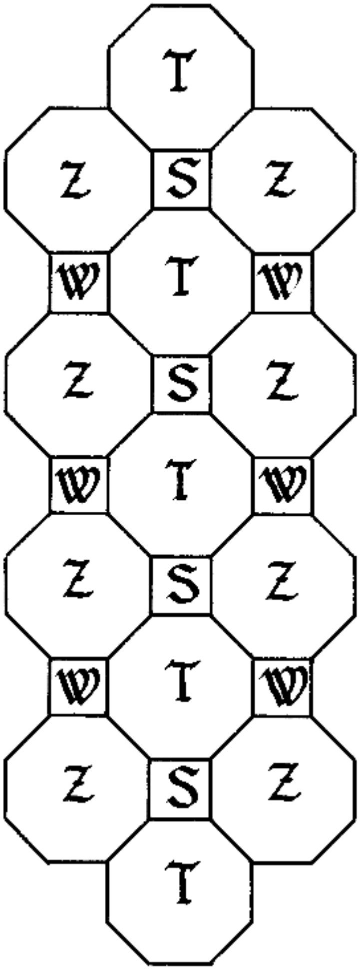

Figure 2. The Larger Board, with Four Types of Place.

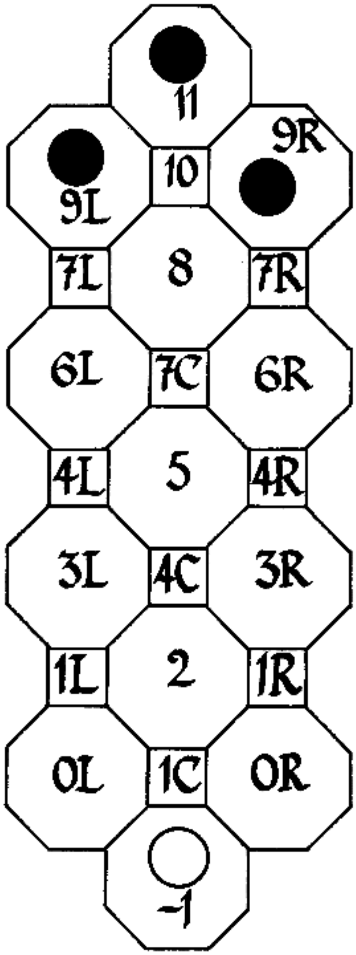

Figure 3. The Larger Board, Numbered for the Trace.

Two Trial Games

If you want to see how the game goes, first set up the board and watch an expert hunter against a novice hare:

hounds:	*abd*	*cbd*	*fbd*	*fed*	*fhd*	*fhg*	*fhj*	*ihj*	(wins)
hare:		*k*	*i*	*j*	*g*	*j*	*i*	*k*	

First game.

The chase looks so easy that the novice decides to direct the hounds in pursuit of an expert hare:

hounds:	*abd*	*cbd*	*fbd*	*fed*	*feg*	*fhg*	*fig*	*eig*	*fig*	*fij*
hare:		*k*	*j*	*i*	*h*	*k*	*j*	*k*	*j*	*k* *h*

Second game.

and now the hare will escape by e or f.

If expert hounds chase an expert hare on Fig. 1, who wins? And what if the hare makes the first move? Or starts from a different place? (See the Extras.) And (when you've become more expert) what about Fig. 3?

History

According to Lucas the game (on Fig. 1) was popular among French military officers in the nineteenth century. Some say it was invented by Louis Dyen; others attribute it to Constant Roy. It was solved by Lucas (1893) and Schuh (1943) and popularized (again) by Martin Gardner (1963). Schuh's analysis was based on a list of 18 classes of winning positions for the hounds (reproduced in the Extras) and he recognized that "the opposition" plays a key role, but he had no exact definition for it. In a later section we'll give a definition which simplifies the game on Fig. 1 and also allows us to solve that on Fig. 3.

The Different Kinds of Place

Let's look at the board more closely. There are really two types of octagon: central ones (T in Fig. 2) and side ones (Z). There are also two types of square: central squares (S) and side squares (W). Except near the very top or bottom of the figure, each T or Z is next to at least one place of every other type, but each W or S is only next to octagons, T and Z. Since W and S are never adjacent, it's sometimes convenient to lump them together into a single class, N. Of the three types T, Z and N, every place, even the ones at the top and bottom, is next to at least one place of each other type, but to none of its own type. The letters correspond to remainders after division by 3 of the numbers from Fig. 3:

Remainder Zero :	Z
Remainder oNe :	N = Weak or Strong
Remainder Two :	T

In Fig. 3 the difference of two numbers in adjacent places is always 1 or 2.

The sum of the numbers occupied by the four animals is an important property of the position; we call it the **trace**. Every move changes the trace by 1 or 2. If the hounds succeed in trapping the hare at the bottom of the board, then the hounds are at 0, 1, 0 against the trapped hare at −1 and the trace is 0. If instead the hounds trap the hare on the side of the board, say at 1L, then the hounds end on 3, 2, 0 against the hare on 1, and the trace is 6. It can easily be checked that

No matter where
You trap the hare,
The trace you'll see
Divides by *three*

TRIALITY TRAPS!

The Opposition

The best way for the hounds to make their trap is to move so that they leave the trace a multiple of 3 at every turn. We call this "keeping the opposition". If they do this, the hare's move must be to a non-multiple of 3, because it changes the trace by 1 or 2. But whenever the trace is not divisible by 3 the hunter usually has a choice of several hound moves which restore it to a multiple of 3, and among these he should find one that restores a winning position.

Threefold traces
Win most chases.

KEEPING THE OPPOSITION

If you check the traces for our first game, with the board numbered as in Fig. 4, you'll see that the hounds always kept the opposition:

hare:	k	i	j	g	j	i	k
hounds:	cbd	fbd	fed	fhd	fhg	fhj	ihj
trace:	9	9	6	6	3	3	0

Since the hare doesn't want to be trapped, he doesn't want the hunter to move to positions whose trace is divisible by 3. The best way to prevent this is for the hare to grab the opposition by moving to such a position himself. Then any hound move will change the trace to a nonmultiple of 3 and the hare is likely to be able to regrab the opposition. This is the way the hare won our second game. The hounds blundered on their second move by playing from 4 to 2, giving a trace of 8, and from then on the hare managed to retain the opposition at every turn:

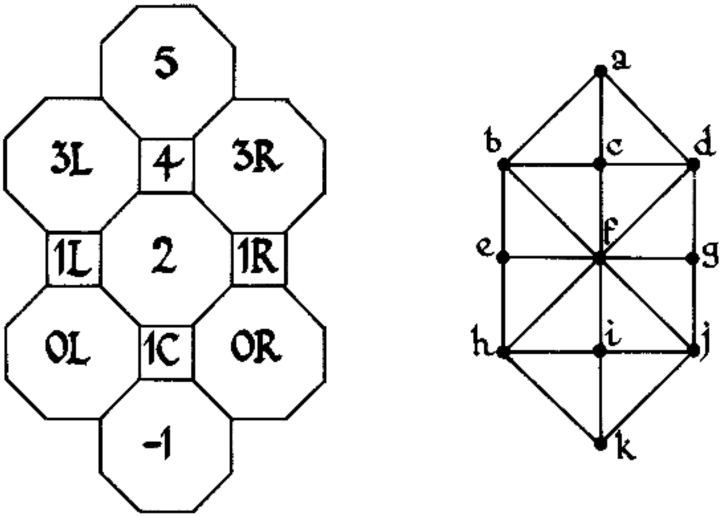

Figure 4. The Board Numbered for Determining the Opposition.

hounds:	*abd*	*cbd*	*fbd*?	*fed*	*feg*	*fhg*	*fig*	*eig*	*fig*	*fij*
hare:	*k*	*j*	*i*!	*h*	*k*	*j*	*k*	*j*	*k*	*h*
trace:	10	10	**9**	6	3	3	3	3	3	3

So whoever can move to a position whose trace divides by 3 is said to have the opposition. The opposition is certainly a valuable commodity which both players desire. But it's not all there is to the game, because sometimes the *hounds* may have the opposition but be unable to keep it without letting the hare escape behind them. In other cases the *hare* may have the opposition for several moves, but then lose it because the hounds block his only moves to places which would restore it. However, such positions are rather rare, and the average player who combines the principle with a little commonsense will usually trap a novice hare on the small board. An annotated example appears on p. 716.

When Has the Hare Escaped?

He has **escaped** if he has passed or is passing two hounds, unless he is on a *square* place (W or S) and the hounds can immediately occupy the neighboring octagons (Z or T) aside or ahead of him.

Although he may have not escaped, the hare is **free** in some other positions in which the hounds can never force him to retreat. This certainly happens if he's strictly passed a hound and is not on a Weak (W) square, or, if he's on a central octagon (T) and is past or passing at least one hound.

Third Game

Hounds	Hare	Trace	Comments
3L, 5, 3R	−1	10	
3L, 4, 3R		9	Taking the opposition
	0R	10	
2, 4, 3R		9	A novice hunter might have moved 4 to 2, giving a "solid" position, but losing the opposition.
	1C	10	
2, 3L, 3R		9	The other "reasonable" move, 3R to 1R, changes the trace by the wrong amount. Since the move from 2 to 1 would allow the hare to escape, there's really only one choice.
	−1	7	
1C, 3L, 3R(!)		6	Because the hounds can't retreat, they can never increase the trace by 2, so to gain the opposition they must decrease 7 to 6 by moving a hound from 2 to 1. A move to 1R or 1L won't lose, but wastes time, since the hare can force the hounds back to the present position position by going to 1C.
	0R	7	
1C, 2, 3R		6	The other two moves (3R to 2, 1C to 0L) that restore the trace to 6 would let the hare escape.
	−1	5	
1C, 2, 4(!)		6	Once again, the other moves (3 to 1, 2 to 0) keeping the opposition would let the hare escape, leaving only this unlikely looking move.
	0R	7	
0L, 2, 4(!)		6	4 to 3R repeats; 2 to 1 allows escape; only 1C to 0L makes progress
	1R	7	
0L, 2, 3R		6	Obvious
	0R	5	
0L, 2, 1R		3	
	−1	2	Hare's last gasp.
0L, 0R, 1R		0	The novice hunter might now lose by playing from 1R to 0R.
	1C	2	
0L, 0R, 2		3	The only time the hounds reach a trace larger than their previous one.
	−1	1	
0L, 0R, 1C		0	Wins.

Losing the Opposition

To analyze the exceptional positions, when someone wins in spite of not having the opposition, it's best to consider the types of place the animals occupy. For example, all the positions where the hounds have just won are of type Z^2NT, meaning that 2 animals are on Z places, 1 on N and 1 on T.

Some of the exceptional cases arise from the difference between the Strong and the Weak types of N places. Each Strong (central) N square is next to *four* other places, while each Weak (side) square is next to only three. Other things being equal, an animal should prefer a Strong place to a Weak one, since both make the same contribution to the opposition; but the Strong place is likely to offer him more choices later. For example, one exceptional case arises when the hounds move to Fig. 5. Despite the fact that the hounds have the opposition

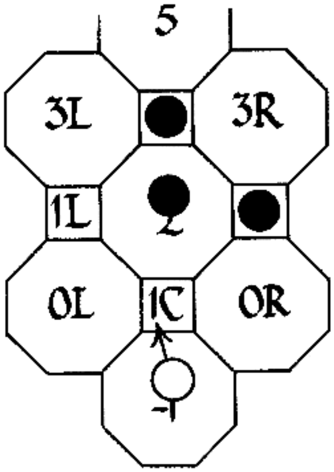

Figure 5. An Exceptional Hare and Hounds Position.

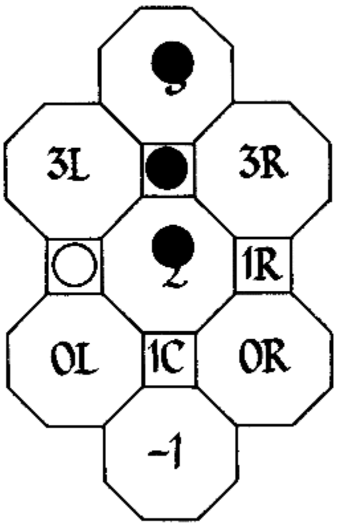

Figure 6. Another Exception to the Opposition Principle.

(trace 3), the hare wins by playing to 1C, because now the only hound moves which keep the opposition let the hare escape. In some sense this N^2T^2 position loses because the hound at 1R is on a Weak square. On the other hand, we saw in our third game that a hare on −1 has no defence against hounds on 4C, 2, 1C (another N^2T^2 position). Unless the hare has passed one or more hounds, S^2T^2 wins for the hounds, but SWT^2 often loses.

As another example, suppose the *hare* has just moved to the position of Fig. 6. He has the opposition, but after the hound on 4C moves to 3L, the hare must retreat to 0L, losing the opposition and the game. But a hare in place 1C against these hounds would have both the opposition and a winning position. Once again, the difference between a Strong and a Weak square means the difference between winning and losing, this time for the hare.

A Strategy for the Hare

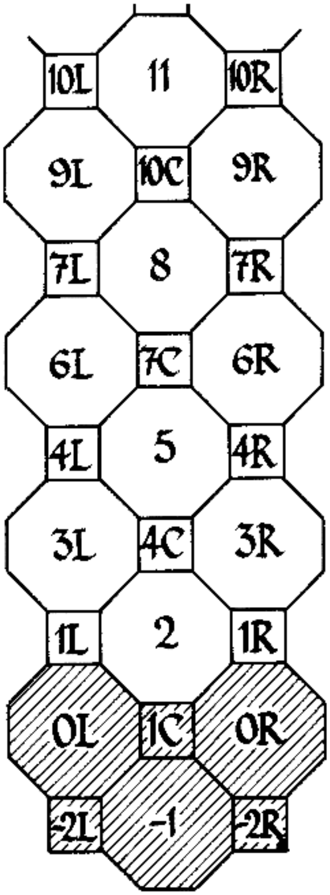

Figure 7. Keeping the Opposition on a Semi-infinite Board.

We'll show that an expert hare that has the opposition on the semi-infinite board of Fig. 7 can either keep it indefinitely or escape, unless he has to start from the **Scare'm Hare'm** position (Fig. 8). In fact the hare will always stay on the six shaded places numbered 1C, 0L, 0R, −1, −2L and −2R, unless the hounds let him out. His basic strategy is to keep the opposition.

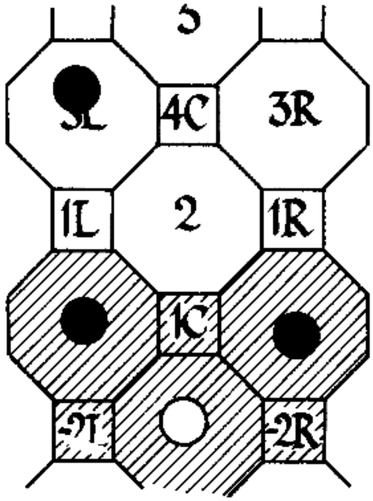

Figure 8. The Scare'm Hare'm Position.

If possible, escape or gain your freedom!
Otherwise, keep on the six shaded places, and
if you can keep the opposition by a move to a
non-Weak place, do so.
If a move to S(1C) is blocked, then
(A) against hounds on T^2S, move to W(−2L or −2R),
(B) against hounds on ZN^2, advance to Z (losing the
opposition) on the other side of the
board from the hound-occupied Z.
If a move to Z (0) is blocked,
(C) go to T (−1) (losing the opposition).

THE HARE'S STRATEGY

If these rules allow two or more moves, choose any one. If they allow none, resign (or hope for a mistake)!

First we show that if the hounds reach Fig. 8, a recent hare's move must have been of type (A), (B) or (C). For if the hounds came from a position in which they *had* the opposition, then the hare, after his last move, *didn't*, and the present position must have been reached by (B) or (C). Otherwise the hounds have come from a position whose trace was congruent to 1, mod 3, and hence from Z^2N, since they are on Z^3 in the figure. At the hare's last move 2 was vacant and either 0L or 0R was occupied by a hound. But if the hare came from 0L, 1C or 0R he could have escaped by moving to 2 and so he must have come, from the Weak square −2, which he can only have reached by a move of type (A).

Suppose you've just made a move of this strategy which was not of type (A), (B) *or* (C). Then you *have* the opposition and you're *not* on a Weak square and the table below shows that the Hare's Strategy always gives you another move, unless you're faced with the Scare'm Hare'm Position.

From \ To	Z	S	T
Z	—	(A) or gain freedom by advance to T.	already free
S	escape by advance to T.	—	already free.
T	escape, since *not* Fig. 8.	(A) or (B)	—

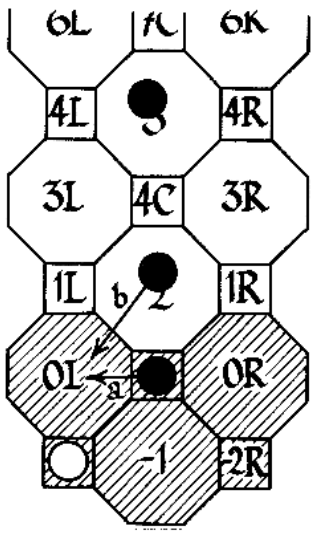

Figure 9. Position after a Move of Type (A).

Next suppose you've just made a move of type (A). Then in the next few moves you can either escape or regain the opposition by a move *not* of type (A), (B) or (C), and from which the hounds can't immediately move to Fig. 8. This is because when (A) is applied, the Strong square 1C must be occupied and also two central octagons (not including −1 because the hare has not escaped); see Fig. 9. Now the only way the Hare's Strategy can lose the opposition from an N square is by a move of type (C) after a hound moves to 0L. But after move (a) in Fig. 9 the hare regains the opposition, while after move (b) he soon escapes. The hounds can't reach the Scare'm Hare'm Position in time.

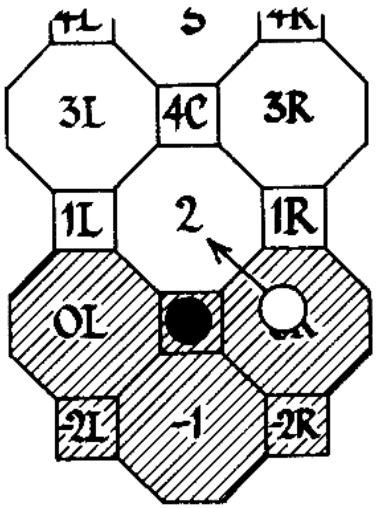

Figure 10. Position after a Move of Type (B).

Now suppose you've just made a move of type (B) (Fig. 10). Then you threaten to escape by moving to the empty T place ahead of you. If the hounds fill this from N, you escape by advancing to W, and if a hound from Z fills it you can reacquire the opposition by retreating to T. The hounds can't straight away reach the Scare'm Hare'm Position.

Finally, *if you've just made a move of type* (C), and were on a Strong square, *both* adjacent Z's must be occupied and you could have escaped. So you were on a Weak square and we have already discussed the situation following your previous move, which must have been of type (A).

On the Small Board

The Hare's Strategy shows that if they don't have the opposition the hounds can only win on the small board by keeping a hound on 5 until they can grab the opposition by moving him to 4 or 3. If they move first from 3L, 5, 3R the hounds can beat a hare starting anywhere except 4. Here is a sample game.

Hounds	Hare	Remarks
3L, 5, 3R	1C	(Or the hare could start on 1L or 1R.)
3L, 5, 2		
	−1	If instead to 0, the hounds take the opposition by moving from 5 to 4.
1L, 5, 2		
	1C	If instead to 0, the hounds take the opposition by moving from 5 to 3.
0L, 5, 2		
	−1	If instead to 0, the hounds take the opposition by moving from 5 to 4.
1C, 5, 2		
		Now, since there is no place −3 on this board, the hare is forced to give the hounds the opposition and the game.

On the Medium and Larger Boards

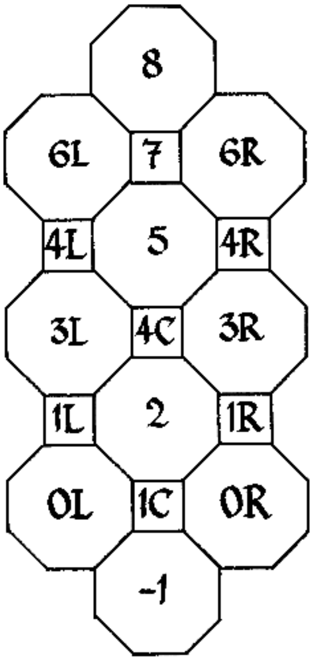

Figure 11. The Medium Board.

By a slight extension of this argument, the hounds, moving from 6L, 8, 6R on the Medium Board (Fig. 11) can trap a hare starting on −1, 0, 2, 3 or 5. Since they have the opposition they can certainly win on the Small Board got by dropping numbers −1, 0 and 1 (Fig. 1). The hare on 2 may reach one of the positions of Fig. 12, forcing the hounds to give him the opposition in return for his retreat, but it is too late, since the hounds can play to 3L, 5, 3R, which wins for them, even without the opposition, because places numbered −2 are not on the board. What if the hare now goes to 0L? See the Extras.

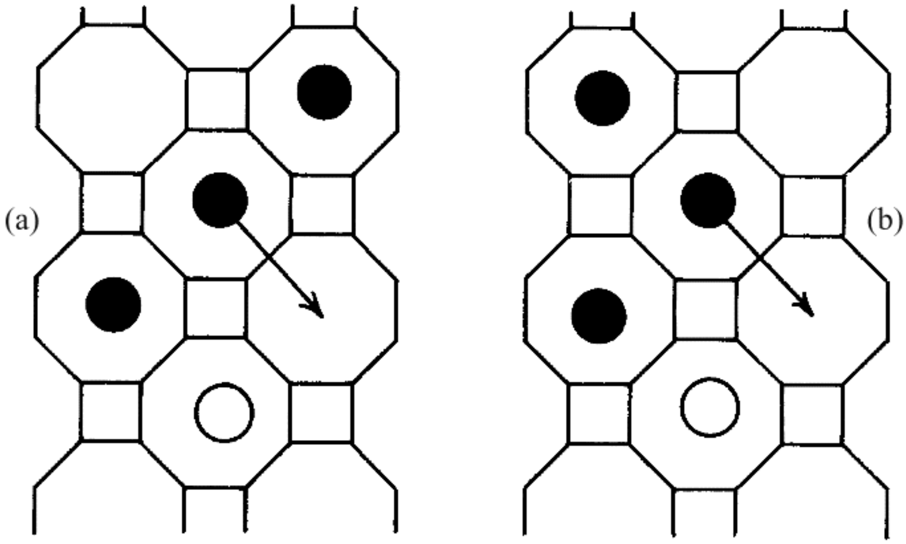

Figure 12. A Sound Bound for a Hound?

It is interesting that the hounds win if the configuration of Fig. 12(a) occurs at 3L, 5, 6R against 2, but not if it is higher (on the Larger Board of Fig. 3) at 6L, 8, 9R against 5. After the hound moves from 8 to 6, the Hare snatches the opposition by retreating to 3 and then follows his Strategy, but using the next set of six squares (4C, 3L, 3R, 2, 1L and 1R) up the board.

It should now be clear that the Hare's Strategy can be improved. If the Hare doesn't have the opposition, he should try to reach a position like 5 against hounds on 6L, 8, 9R (all such positions have trace 28). The way to force the hounds to move into such a position is to move to one whose trace is larger than the desired one by a small multiple of 3. In fact we can prove that

> on the Larger Board (Fig. 3) the hounds can win from a position of trace 31 only if the Hare is on a Weak square or the position is 6, 10, 11 *versus* 4C (Fig. 13).

THE THIRTY-ONE THEOREM

The proof is sketched in the Extras.

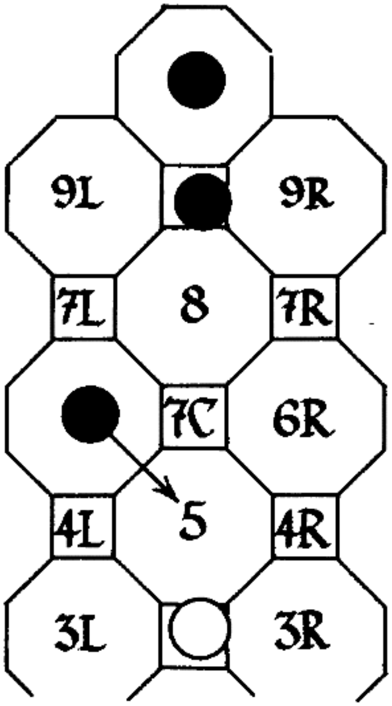

Figure 13. The Hound-Dog Position.