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LOGIC EDUCATION:
Playing with True and False

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Affidavit

I, undersigned, Luigi Bernardi, hereby declare that the work presented in this manuscript is my own work, carried out under the scientific direction of Emmanuel Beffara, Myriam Quatrini, and Lorenzo Tortora de Falco, in accordance with the principles of honesty, integrity and responsibility inherent to the research mission. The research work and the writing of this manuscript have been carried out in compliance with both the French national charter for Research Integrity and the Aix-Marseille University charter on the fight against plagiarism.

Fait à Marseille le 04/04/2024

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Liste de publications et participation aux conférences

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2. Bencivenni, I., Bernardi, L., Ferretti, F., & Tomasi, L. (2022). Fregi e tassellazioni del piano per guardare la realtà che ci circonda con occhio matematico. In *Apprendimento laboratoriale in Matematica e Fisica in presenza e a distanza, Torino, 11-12-13 ottobre 2021*. Università degli Studi di Torino.

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5. Bernardi, L. (2024). Avac e afru: i nostri studenti sanno distinguere vero e falso? *Insegnamento Matematica e Scienze Integrate*, Vol 47(A), n.1, gennaio 2024.

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Résumé

Cette thèse introduit une interprétation dialogique des preuves à travers le jeu $\mathbf{T}\mathscr{UVA}$, en s'appuyant sur le travail de Krivine and Legrandgérard (2007). En particulier, elle montre la correspondance entre les stratégies gagnantes dans le jeu et les preuves dans LK. Elle explore ensuite l'application de ce cadre théorique dans la didactique des mathématiques, en analysant les connexions entre les concepts de preuve en logique mathématique et en didactique, à travers les lentilles théoriques fournies par le jeu $\mathbf{T}\mathscr{UVA}$. Un programme éducatif pour introduire à la logique, basé sur le jeu et aligné avec les théories de la didactique des mathématiques, est alors proposé et analysé.

Mots-clés : logique, didactique des mathématiques, théorie des jeux, sémantique des jeux, théorie de la démonstration

Sunto

Questa tesi introduce un'interpretazione dialogica delle dimostrazioni attraverso il gioco $\mathbf{T}\mathscr{UVA}$, basandosi sul lavoro di Krivine e Legrandgérard (2007). In particolare, viene mostrata la corrispondenza fra strategie vincenti nel gioco e dimostrazioni in LK. Viene poi esplorata l'applicazione di questo quadro teorico nella didattica della matematica, analizzando le connessioni fra i concetti di prova in logica matematica e in didattica, attraverso le lenti teoriche fornite dal gioco $\mathbf{T}\mathscr{UVA}$. Si propone e si analizza quindi un programma educativo di introduzione alla logica, fondato sul gioco e in linea con teorie di didattica della matematica.

Parole chiave: logica, didattica della matematica, teoria dei giochi, semantica dei giochi, teoria della dimostrazione

Abstract

This thesis introduces a dialogical interpretation of proofs through the game $\mathbf{T}\mathscr{UVA}$, building on the work of Krivine and Legrandgérard (2007). In particular, it shows the correspondence between winning strategies in the game and proofs in LK. It then explores the application of this theoretical framework in Mathematics Education, analyzing the connections between the concepts of proof in logic and in education, through the theoretical lenses provided by the game $\mathbf{T}\mathscr{UVA}$. It proposes and analyzes an educational program for an introduction to logic, founded on the game and in line with theories of mathematics education.

Key Words: Logic, Mathematics Education, Game Theory, Game Semantics, Proof Theory

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Contents

List of Figures

List of Tables

Introduction

Mathematics is an opinion.

Mathematics is an opinion on a historical level, both regarding events and the analysis of the intentions behind them.

Mathematics is an opinion on a social level, both from an institutional point of view—based on the importance that politics attributes to mathematics—and from a practical needs perspective.

Mathematics is an opinion on an educational level, both in the debate on which content, methodologies, and theoretical frameworks are most suitable for Mathematics Education, and for the undeniable difference in beliefs and knowledge of its interpreters[1].

Mathematics is an opinion on a theoretical level. Just consider how the concept of a proof, which in theory should represent the pinnacle of indisputability, has changed significantly over the centuries. Consider for example, in geometry, how what was regarded as rigorous according to Euclid (in the "Elements") was not rigorous according to D. Hilbert (1899) and what was considered rigorous according to Hilbert was not rigorous according to A. Tarski (1959), at least at a language level. Consider also the words with which Euler, in his *Elementa Doctrinae Solidorum* (1758), concludes the proof of the Polyhedral Formula: "Cum igitur veritas propositionis in his omnibus casibus sibi constet, dubium est nullum, quin ea in omnibus omnino solidis locum habeat, sicque propositio sufficienter videtur demonstrata"[2]. A famous discussion over the soundness of a proof concerns the Four Color Theorem. In 1976, Appel and Haken (1976) succeeded in reducing the infinitely many maps to finitely many ones, but too many to be directly checked by a human being. These 1834 configurations were checked to be effectively four colorable by a computer (it took around a thousand hours to do the computing). This son-of-times proof procedure, which heavily relies on the correct behaving of the computer, was not accepted by all in the mathematical community.

Beyond these considerations, which mainly focus on the human perspective and the conventions adopted by the mathematical community, the opinability of mathematics on a theoretical level can reach even greater extremes: notably, the consistency of some theories is questionable within the same theories[3].

[1]"Pour des élèves de cinquième la somme des angles d'un triangle ne peut être égale à 180° pour tout triangle, parce qu'un petit triangle ne peut avoir même somme d'angles qu'un triangle plus grand"(Balacheff 1987)

[2]Since the truth of the statement is confirmed in all these cases, there is no doubt that it holds absolutely in all solids, and thus the statement seems sufficiently demonstrated.

[3]Just think of the classic Gödel's incompleteness theorems regarding various theories, including

Nevertheless, in society's eyes, mathematics appears not just strict and uncreative—where there is no room for debate, creativity hardly finds fertile ground—but also as something static and unique, never changed nor changeable, which sits independently from the social and economic fabric in which it is positioned. In other words, mathematics is not perceived as something dynamic, where different opinions can exist and be debated; rather, it is seen as something abstract and procedural, with procedures that often lack any meaning to those who are using them.

I firmly believe that a crucial task of Mathematics Education is to challenge these perceptions and biases, unveiling mathematics as a discipline inherently and profoundly dialogical. The facade of mathematics, with its stony appearance of absolute truths, is nothing but the product of its profound debatability: where everything can be questioned and there are no experiments that can indisputably establish what is true and what is not, one must be ready to 'fend off attacks from all sides', leaving no room for chance and specifying precisely what is being said and how, to avoid being misunderstood. Just consider how formal mathematics found fertile ground in contexts such as Ancient Greece, where critical thinking and discussion were central values. It seems, in fact, that the idea of proof was born at that time, seeing "the source of deductive mathematics and logic in dialectical argument and disputation"[4]. It is noteworthy that, in the same period and place, a significant cultural revolution was concurrently unfolding: the three possible forms of government in a city-state (Tyranny[5], Oligarchy, and Democracy) all shared a common principle: power was earned and not given by a superior god (Barbero 2023). The rise to power depended only on one's own ability to persuade others.

A *proof* is the ultimate expression of the opinability of mathematics. After the hours of reflection and critical thinking that go into a proof, one would believe that the result obtained is immune to any contradiction. In other words, it is considered possible to convince anyone of the truth of one's statements. I like to define 'proving' as indeed *the act of convincing a free person.*

However, if these were the premises, 'doing a proof' would have a different meaning from what it typically has in an educational environment. What is often seen in the classroom when 'doing a proof' is, in fact, more akin to 'teaching to be convinced', which is a paradoxical concept. What can be done instead is to teach student to be critical and skeptical, and accustom them to being so, bringing out dialogue as the fountainhead of new knowledge, and establishing the search, through *trials*, of *examples* and *counterexamples* as one of the foundational elements of mathematical and scientific practice.

This dissertation introduces a dialogical interpretation of mathematical proofs, revisiting and expanding upon the work of Krivine and Legrandgérard (2007). The work is grounded in a well-established literary corpus, ranging from the methods of

fundamental ones such as arithmetic and set theory, see for example (Abrusci and Tortora de Falco 2018).

[4]K. Fritz cited by Ernest (1986).

[5]The word 'tyranny' is not meant in a negative sense, it simply refers to the power of a single individual.

analysis and synthesis[6] to a development towards logical-mathematical approaches, attempting to interpret the proof process within the framework of Game Theory and Game Semantics (Abramsky and Mccusker 1999; Hyland and Ong 2000; Laurent 2010; Abramsky, Jagadeesan, and Malacaria 2014). While remaining true to the logical core of the approach, this work draws inspiration from studies such as (Brousseau 1997; Vernant 2007; T. Barrier 2008; Arzarello and Soldano 2019) to explore how the concepts discussed can prove beneficial in Mathematics Education[7].

In Chapter 1, a game between two players called $\mathbf{T}\mathscr{UVA}$ is introduced and analyzed. This game is an extension of the $\mathscr{UVA}$ game presented in (Krivine and Legrandgérard 2007). The core idea of the game is a debate between two players about the truth of a mathematical statement. Before providing the formal definition, an intuitive idea of the game is given and some examples are illustrated.

In Chapter 2, a brief discussion is presented on what a proof in Mathematical Logic is and how the concept has evolved historically. A derivation system is then illustrated, which is subsequently modified through various transformations[8] leading to the system $\mathrm{LK}_{\mathrm{game}}$, a system classically equivalent to LK , but more suited to interpreting dialogical situations.

In Chapter 3, the correspondence, under certain conditions, between derivations in $\mathrm{LK}_{\mathrm{game}}$ and winning strategies in the game $\mathbf{T}\mathscr{UVA}$ is proved.

In Chapter 4, we present the game equivalent of cut elimination, namely the composition of strategies: from two or more winning strategies that interact with each other, a new winning strategy is obtained.

In Chapter 5, the potential epistemological value of the game is discussed. The platform `www.oiler.education/lui` is then introduced, where one can play the game online, exploring various different mathematical theories.

In Chapter 6, the initial discussion focuses on the connections between the concept of proof in Mathematical Logic and in Mathematics Education, through the theoretical lens provided by the game $\mathbf{T}\mathscr{UVA}$. The chapter then addresses some of the challenges encountered at the school level related to proof, advocating for an explicit introduction of logic from the early years of education.

Over the next three chapters, an educational program for introducing logic is proposed. The ideas and activities behind the educational paths created have been guided by suggestions from the literature in Mathematics Education and the formal game $\mathbf{T}\mathscr{UVA}$.

Chapter 7 presents Zermelo, an educational path based on describing and verifying properties related to certain sets. This path pays particular attention to logic, dialogue,

[6]This distinction, already present in its embryonic stage in Aristotle's *Prior Analytics*, was clarified by Pappus (Collectio Matematica, II), and later discussed throughout history several times, by authors such as R. Descartes, G. Polya Polya (1962) and J. Hintikka.

[7]We emphasize that all studies referenced in the field of Mathematics Education are grounded in theoretical works from the 1960s and 1970s (Lorenzen 1961; Hintikka 1976; Lakatos 1976). One question that has driven this research is whether recent developments in Game Semantics might enhance this line of inquiry.

[8]In particular, we are referring here to the *focusing* constraints, first explored in (Andreoli 1992), and to the *reversion* constraint, explored for instance in (Laurent, Quatrini, and Tortora de Falco 2005).

and quantifiers. It also discusses some results that emerged from the classroom experimentation.

In Chapter 8, the Bul path is introduced, where the focus shifts from quantifiers to predicate logic and connectives, with the aim of delving into some linguistic-mathematical aspects.

Chapter 9 presents the Lovleis path, which is divided into two distinct phases: a narrative phase, during which a story in which the class is the protagonist is read to the students, and a playful phase, where two-player games that emerge from the narrative are explored. The focus of the path is the analysis of strategies. The chapter also discusses some results from the experiments, comparing the difficulties students encountered in finding and presenting a strategy in a two-player game with the typical difficulties of proving in mathematics.

1. The Game

In this first chapter, we introduce the $\mathbf{T}\mathscr{UVA}$ game, *a game where two players debate about the validity of a formula.* Specifically, one of the players believes that the formula is true, while the other believes it is false.

The game $\mathscr{UVA}$, initially introduced by Krivine and Legrandgérard (2007), is further developed by incorporating theories ($\mathbf{T}$). A more formal definition is then provided through the use of graphs.

1.1. Preliminary definitions

Before explaining how the game works, it is important to define how to construct a formula, *i.e.*, a statement over which the players will debate.

Intuitively, a formula F is an expression of the form $\forall\vec{x}(F_1(\vec{x}),\ldots,F_n(\vec{x}) \to A(\vec{x}))$, which should be understood as follows: if, once fixed all the variables $\vec{x} = (x_1, x_2, \ldots)$, all the hypotheses F_1, ..., F_n hold, then the conclusion A also holds. The definition is recursive, meaning that the formulas F_1, ..., F_n will have the same structure as the formula F (possibly with $n = 0$), while the formula A will always be very simple, without connectives or quantifiers, that is, an atomic formula.

This particular structure of the formulas, known as the Krivine normal form, as we will see shortly, makes the rules of the game very straightforward.

More formally, to be able to define the formulas, we must first define language and terms.

Definition 1 (language)**.** *A language $\mathscr{L}$ consists in two at most countable sets: one of* function symbols *and one of* relation symbols. *Each symbol is assigned an arity, which is a natural number. Symbols of arity* 0 *are called* constants.

In the following, f will identify a generic function symbol, while R a generic relation symbol.

Definition 2 (terms and formulas)**.** *Given a language $\mathscr{L}$, terms and formulas over $\mathscr{L}$ are defined inductively as follows:*

Each variable is a term. If $t_1,\ldots,t_n$ are terms then $f(t_1,\ldots,t_n)$ is a term[1].

$$\textit{terms} \qquad\qquad t ::= x \mid f(t_1,\ldots,t_n) \qquad\qquad \textit{where } n \textit{ is the arity of } f$$

[1] Clearly, the generic function symbol could be a constant, meaning $n = 0$.

Every relation between terms is an atomic formula. If F and G are formulas, $F \to G$ and $\forall x\, F$ are formulas.

atomic formulas	$A ::= R(t_1, \ldots, t_n)$	*where n is the arity of R*
general formulas	$F, G ::= A \mid F \to G \mid \forall x\, F$	

where x is a variable. The quantifier $\forall x$ serves as a binder for x, with formulas being considered up to the renaming of bound variables. We represent the complexity of the formula F by $\mu(F)$, signifying the total number of connectives and quantifiers present. A formula is called a closed formula if all its variables are bounded.

Let us notice that, for the logical part of the language, we restrict to the minimal system with only $\to$ and $\forall$ as these allow for a very synthetic yet expressive system. Indeed, it can be proven that, if we have a nullary relation symbol $\perp$ to represent false, then every connective can be expressed in this language, up to classical provability.

We identify the even stricter fragment of *normal formulas*.

Definition 3 (Krivine normal formula)**.** *Normal formulas are defined inductively as follows: if $x_1, \ldots, x_k$ are pairwise distinct variables, $F_1, F_2, \ldots, F_n$ are normal formulas and A is an atomic formula then $\forall x_1 \cdots \forall x_k (F_1 \to (F_2 \to \cdots (F_n \to A)))$ is a normal formula.*

For a normal formula F, following these notations, k is called the arity of F and is written $\mathrm{ar}\, F$*, n is called the degree of F and is written* $\deg F$*.*

Because of the structure of normal formulas, we will consider that the connective $\to$ associates to the right and has priority over $\forall$, besides we use the vector notation $\vec{x}$ to represent finite sequences of terms, so that the general form of a normal formula can be written $\forall \vec{x}\, F_1, \ldots, F_n \to A$ with no parentheses. This notation should be read in the following way: for every $\vec{x}$, if all the premises F_1, ..., F_n are true, then the conclusion A is also true.

Given a normal formula $F = \forall x_1 \ldots x_k\, F'$ and a sequence of terms $\vec{t} = t_1, \ldots, t_k$, we write $F(\vec{t})$ to represent the formula $F'[t_1/x_1, \ldots, t_k/x_k]$. Morover, given an unquantified normal formula $F = F_1, \ldots, F_{\deg F} \to F_0$, we index its subformulas as F_i.

These notations allow us to write $F(\vec{t})_i$ to refer to the i-th premiss (or conclusion if $i = 0$) of F instantiated with terms $\vec{t}$ to replace the variables $\vec{x}$.

Lemma 1. *Each formula F written in our language is equivalent up to classical provability to a formula $\hat{F}$ which is in normal form.*

Proof. The proof proceeds by induction on the complexity of the formula. If F is atomic then $\hat{F} = F$. If $F = \forall \vec{x} G$ with G in normal form then, as before, $\hat{F} = F$. Let us now consider the last case where $F = G \to H$ with G and $H = \forall \vec{x} H_1, \ldots, H_n \to H_0$ in normal form. The initial step is to ensure that the bounded variables in H do not appear in G. A variable that is not in a formula is said to be *fresh* with respect to that formula. This can be achieved just by renaming variables. The normal form of F is $\hat{F} = \forall \vec{x} G, H_1, \ldots, H_n \to H_0$. As a result byproduct of this proof, we notice that a formula F is closed if and only if $\hat{F}$ is also closed. □

Example 1. *Let R and S be two symbols for unary relations. Let us consider the formula $F = \forall x R(x) \to \forall x S(x)$. The first normal form we could think of is $G = \forall x(R(x) \to S(x))$. This is clearly a normal formula but is not equivalent to F. To understand it we can think of $R(x)$ to express x is even and $S(x)$ to express x is odd. In our interpretation F is true—indeed the antecedent is false so the implication is true—while G is obviously false on all even numbers. Following the procedure given in the previous proof of Lemma 1, the correct normal form of F is $\hat{F} = \forall y(\forall x R(x) \to S(y))$.*

Definition 4 (Theory)**.** *A theory **T** is a set of closed normal formulas.*

1.2. Intuitive Insights into the T$\mathscr{U}\mathscr{V}\mathscr{A}$ Game

Before presenting the formal definition of the game, we prefer to provide a more informal and intuitive approach, complemented by some examples of plays.

The game is played between two players called **P** (for *Proponent*) and **O** (for *Opponent*) over a closed normal formula F and within a theory **T**. The board is composed of four sets **T**, $\mathscr{U}$, $\mathscr{V}$, $\mathscr{A}$ of closed normal formulas. Intuitively, **T** is the set of statements that both players believe to be true[2], $\mathscr{U}$ is a set of statements **O** believes to be true, $\mathscr{V}$ is a set of statements **P** believes to be true and $\mathscr{A}$ is a set of statements both players believe to be false. The game is initialized by setting $\mathscr{U} = \{F \to \bot\}$, $\mathscr{V} = \{F\}$ and $\mathscr{A} = \{\bot\}$. In other words, **P** asserts that F is true, **O** that it is false, and both players agree that $\bot$ is indeed false.

The idea behind the game is that, at each turn, the players must falsify a formula chosen from the set of formulas that the other player considers to be true. To falsify a formula $F = \forall \vec{x} F_1, \dots, F_n \to A$, means to find closed terms $\vec{b}$ such that, even though $F(\vec{b})_1$, ..., $F(\vec{b})_n$ are all true, $F(\vec{b})_0$ is false. By iterating this dynamic, the statement is split down into increasingly simpler statements, thanks to the normal form shape.

More precisely, the two players must alternate following these rules, and the first to play is **O**.

- **O** plays by choosing a formula $F \in \mathscr{V}$ and closed terms $\vec{b}$ to be substituted to variables $\vec{x}$ of the top-level quantifiers of F. They add $F(\vec{b})_1, \dots, F(\vec{b})_n$ to $\mathscr{U}$ and $F(\vec{b})_0$ to $\mathscr{A}$. In particular, if $\mathscr{V} = \emptyset$ then **O** cannot move.

- **P** plays by choosing a formula $F \in \mathscr{U} \cup \mathbf{T}$ and closed terms $\vec{b}$ such that $F(\vec{b})_0 \in \mathscr{A}$. They replace the set $\mathscr{V}$ with $\{F(\vec{b})_1, \dots, F(\vec{b})_n\}$.

As can be observed, there is an asymmetry in the moves. While **O** can add atomic formulas to $\mathscr{A}$ (*i.e.*, the set of statements that both players believe to be false), **P** can only play a formula if it is "justified" by a previous move of **O**. Conversely, only **P** can play formulas of **T** (*i.e.*, the set of statements that both players believe to be true). Another asymmetry is that while the set $\mathscr{V}$ is renewed each time, $\mathscr{U}$ and $\mathscr{A}$ are

[2] That is, the set of axioms of the theory.

increasing during a play. Moreover, given the structure of normal formulas, $\mathscr{A}$ will contain atomic formulas only.

Each player can resign instead of playing a move. If the game is finite, the winner is the last player to move. If the game is infinite, then **O** is the winner. However, since the game is—as mentioned—classically initialized by setting $\mathscr{U} = \{F \to \bot\}$, $\mathscr{V} = \{F\}$, and $\mathscr{A} = \{\bot\}$, excluding the possibility of surrendering, **P** will always be able to play $F \to \bot$ (since $\bot \in \mathscr{A}$), which means restarting the game. If the initial position is the one mentioned, the only way for **O** to win is to make the play infinite. On the other hand, **P** wins by emptying the set V, that is, by playing a formula that has no premises: at that point, **O** cannot move and loses the game.

We now provide some examples of plays in different theories. Changing the theory means changing the set **T** and the language with which the formulas can be written. If the environment is propositional logic or pure first-order logic, clearly $\mathbf{T} = \emptyset$.

To gain a deeper understanding of the game, it is suggested that the reader—after reading the following examples—explore the online implementation available at `www.en.oiler.education/lui`. A guide to the online implementation can be found in Section 5.2.

Example 2 (Propositional Logic)**.** *Let us start with an example in propositional logic, where the rules are simplified due to the absence of quantifiers.*

In particular, let us play in the Pierce formula $F = ((P \to Q) \to P) \to P$*. This formula, like every other formula in propositional logic expressed in our language, is already in normal form.*

The game is not very engaging on a strategic level as every **O** *move is forced. However, we hope it can provide an initial insight into what occurs in the game.*

A possible play is illustrated in Figure 1.1. Observe that, at any given line in Figure 1.1, the sets $\mathscr{U}$ *and* $\mathscr{A}$ *actually contain all the formulas in the associated column from the top to the considered line, while* $\mathscr{V}$ *includes only the formulas in the current line.*

Clearly, the move indicates the formula chosen by the player in that round. Remember that, being in a propositional logic environment, players don't choose closed terms during their turn.

The play ends when **O** *is no longer able to make a move, resulting in* **P** *being the winner.*

turn	$\mathscr{U}$	$\mathscr{V}$	$\mathscr{A}$	move
O	$F \to \bot$	F	$\bot$	F
P	$(P \to Q) \to P$	–	P	$(P \to Q) \to P$
O		$P \to Q$		$P \to Q$
P	P	–	Q	P
O		$\emptyset$		`cannot move`

Figure 1.1.: Example play for the Pierce's formula

Example 3 (False formula in pure first-order logic)**.** *As a second example, let us consider a formula with little logical interest, but that nonetheless helps in illustrating the dynamics of the game:* $F = \forall x A(x) \to B(x)$[3]*. The formula is clearly false, and thus, it is expected that the Opponent will manage to win, i.e. the play will be infinite. As we see in Figure 1.2, the Proponent can not play* $A(y)$*, since* $A(y) \notin \mathscr{A}$*. At each turn, the only move they can do is to play* $F \to \bot$ *to restart the game. The play will be thus infinite.*

turn	$\mathscr{U}$	$\mathscr{V}$	$\mathscr{A}$	move
O	$F \to \bot$	F	$\bot$	F, y
P	$A(y)$	–	$B(y)$	$F \to \bot$
...				

Figure 1.2.: Example of a infinite play where **P** can do nothing but restart the game.

Example 4 (Drinker's formula)**.** *The standard formulation of the Drinker's fromula is* $\exists x (Dx \to \forall y Dy)$ *which is clasically equivalent to the normal formula* $F = (\forall x (\forall y Dx \to Dy) \to \bot) \to \bot$*. Figure 1.3 shows an example play on this formula; recall that, at any given line of Figure 1.3, the sets* $\mathscr{U}$ *and* $\mathscr{A}$ *actually consist in all the formulas in the associated column from the top to the considered line, while* $\mathscr{V}$ *consists in the formulas in the current line only.*

P *manages to win the play. However, they have to pick twice a same formula in* $\mathscr{U}$*.*

turn	$\mathscr{U}$	$\mathscr{V}$	$\mathscr{A}$	move
O	$F \to \bot$	F	$\bot$	F
P	$\forall x(\forall y Dx \to Dy) \to \bot$	–		$\forall x(\forall y Dx \to Dy) \to \bot, t$
O		$\forall y Dt \to Dy$		$\forall y Dt \to Dy, u$
P	Dt	–	Du	F_1, u
O		$\forall y Du \to Dy$		$\forall y Du \to Dy, v$
P	Du	–	Dv	Du
O		$\emptyset$		`cannot move`

Figure 1.3.: Example play for the Drinker's formula $F = (\forall x (\forall y Dx \to Dy) \to \bot) \to \bot$

In interpreting Figure 1.3, it is important to remember that $\to$ takes precedence over $\forall$.

Example 5 (PA theory)**.** *As a final example, let us discuss a formula in Peano's theory. Although we will delve into the theoretical aspects more thoroughly in Chapter 5, we give the reader a preview that the formula is written with predicates whose descriptions are*

[3]Recall, as mentioned above, that $\to$ has precedence over $\forall$

*contained in **T**. For example, regarding the predicate* $>$, $DEF_{>} = x > y \iff \exists z(y+z=x) \in \boldsymbol{T}$. *The equivalent normal form of* $\exists z(y+z=x) \in \boldsymbol{T}$ *is* $\forall z(y+z=x \to \bot) \to \bot$.

Consider the formula, already in normal form, $F = \forall x\, (\forall y\, (PRIME(y),\ y > x \to \bot) \to \bot)$*; which asserts that prime numbers are infinite.*

In the play in Figure 1.4, **O** *asserts that there are no prime numbers greater than* 25. **P** *counters by saying that* 29 *is both prime and greater than* 25*. At this point, to continue the discussion,* **O** *would have to deny one of the two statements between* $PRIME(29)$ *and* $29 > 25$*, placing it in* $\mathscr{A}$*. Let us say that* **O** *affirms that* $29 > 25$ *is false. To be able to counter in turn,* **P** *must now play the definition of the predicate* $>$ *that is present in **T**, showing how—according to the definitions both agree upon—*29 *is indeed greater then* 25*. Indeed* $29 = 25 + 4$.

A play thus wll simulates a process where what was initially stated is progressively clarified.

turn	$\mathscr{U}$	$\mathscr{V}$	$\mathscr{A}$	move
O	$F \to \bot$	F	$\bot$	F, 25
P	$\forall y(\mathrm{PRIME}(y), y > 25 \to \bot)$	–		$\forall y(\mathrm{PRIME}(y), y > 25 \to \bot)$, 29
O		$\mathrm{PRIME}(29), 29 > 25$		$29 > 25$
P		–	$29 > 25$	$DEF_{>}$, 29, 25
O		$\forall z((25+z=29) \to \bot) \to \bot$		$\forall z((25+z=29) \to \bot) \to \bot$
P	$\forall z((25+z=29) \to \bot)$	–		$\forall z((25+z=29) \to \bot)$, 4
...				

Figure 1.4.: Example play on the statement that primes are infinitely many.

1.3. Formal Definition of the T$\mathscr{U}\mathscr{V}\mathscr{A}$ Game

The definition of game previously discussed is an intuitive one, aimed at providing the reader with the opportunity to understand the dynamics of the game. We now state the formal definitions for the structure of the game. In all statements below, we assume that a first-order language and a theory **T** over it are given. Players are identified by two symbols **P** and **O**. If p designates a player, we call $\neg p$ the other player.

Definition 5. *Let* $\mathbf{G}_{\boldsymbol{T}}$ *be the bipartite directed graph whose vertices are split in to*

- $\mathscr{P}_{\mathbf{P}}$*, the* **P***-positions, which are pairs* $(\mathscr{U}, \mathscr{A})$ *where* $\mathscr{U}$ *is a finite set of closed normal formulas and* $\mathscr{A}$ *is a finite set of closed atomic formulas,*
- $\mathscr{P}_{\mathbf{O}}$*, the* **O***-positions, which are triples* $(\mathscr{U}, \mathscr{V}, \mathscr{A})$ *where* $\mathscr{U}$ *and* $\mathscr{V}$ *are finite sets of closed normal formulas and* $\mathscr{A}$ *is a finite set of closed atomic formulas,*

and whose edges (called moves) are defined as follows:

- *a* **P***-move from a position* $(\mathscr{U}, \mathscr{A})$ *is a pair* $(F, \vec{b})$ *where* $F \in \mathscr{U} \cup \boldsymbol{T}$ *and* $\vec{b}$ *is a sequence of closed terms such that* $F(\vec{b})_0 \in \mathscr{A}$*, its target is the position* $(\mathscr{U}, \{F(\vec{b})_1, \ldots, F(\vec{b})_{|F|}\}, \mathscr{A})$*;*

- *an* **O***-move from a position* $(\mathscr{U}, \mathscr{V}, \mathscr{A})$ *is a pair* $(F, \vec{b})$ *where* $F \in \mathscr{V}$ *and* $\vec{b}$ *is a sequence of closed terms, its target is the position* $(\mathscr{U} \cup \{F(\vec{b})_1, \ldots, F(\vec{b})_{|F|}\}, \mathscr{A} \cup \{F(\vec{b})_0\})$*.*

Definition 6 (play)**.** *A play is a finite or infinite walk in the graph* $\mathbf{G}_{\boldsymbol{T}}$*.*

We note that not only in the formal definition is the game not initialized in the classical way, but also that the general definition of a play allows it to start from a position of either player. We write $a \xrightarrow{\pi}$ to state that play π starts from a position a and $\xrightarrow{\pi} b$ to state that it is finite and leads to a position b. For $a \xrightarrow{\pi_1} b \xrightarrow{\pi_2}$ we write $\pi_1\pi_2$ for a play obtained by concatenation of π_1 and π_2. We identify each move with the play that consists in that single move.

Definition 7 (strategy)**.** *Let p be a player. A p-strategy σ is a set of p-moves with pairwise distinct source positions. A σ-play is a play where all p-moves are elements of σ.*

In other words, a p-strategy is a partial function σ mapping p-positions to valid moves. If a is a position, we will call $\sigma(a)$ the move in σ with source a if it exists.

Remark 1. *This definition corresponds to the usual notion of* positional *or* history-free *strategy. As we will see, this is sufficient to get our results. In some sense, the sets* $\mathscr{U}$ *and* $\mathscr{A}$*, which increase along plays, record just enough history for the game to be well structured.*

1.3.1. Winning Conditions

We now define the winning condition on plays and its associated notions.

Definition 8 (winner)**.** *The winner of a play is* **P** *if the play is finite and ends on an* **O***-position, it is* **O** *otherwise.*

Definition 9 (winning strategy)**.** *A p-strategy σ is* winning from a position *a if all maximal σ-plays that start from a are winning for p.*

The maximality condition is understood among σ-plays only. It simply means that we only consider plays where p actually moves in positions where σ does make a choice and where $\neg p$ always moves if possible. A maximal play is either infinite or finite with a last position from which there is no possible move (respecting σ, if it is a p-position).

Definition 10 (winning position)**.** *A position a is p-winning if there exists a p-strategy that is winning from a.*

If a strategy σ is winning from a position a and a strategy τ is winning from another position b, there is no reason that σ and τ agree on the positions they have in common. Formally, one could even imagine that a single strategy could not be winning for both positions. The following statements establish that it is not the case.

Lemma 2. *Let p be a player. A p-position a is p-winning if and only if there exists a p-move from a to a p-winning position.*

Proof. If σ is a p-winning strategy from a, then $\sigma(a)$ is defined and leads to a $\neg p$-position b. For each maximal σ-play π from b, $\sigma(a)\pi$ is a maximal σ-play from a, so it is winning for p by hypothesis, hence π is also winning for p. Therefore b is p-winning.

Reciprocally, consider a move $a \xrightarrow{m} b$ and suppose there exists a p-strategy σ that is winning from b. If σ is not already winning from a, define the strategy σ' as σ except that $\sigma'(a) = m$. Consider a maximal σ'-play from a. Such a play has the shape $a \xrightarrow{m} b \xrightarrow{\pi}$ where π is a maximal σ'-play. Suppose that π is not a σ-play, then by construction it means that the play decomposes as $b \xrightarrow{\pi_0} a \xrightarrow{\pi'}$ where π_0 is a σ-play. By hypothesis σ is not winning from a so there is a maximal σ-play π'' from a that is $\neg p$-winning, but then $\pi_0\pi''$ is a maximal σ-play from b that is $\neg p$-winning, which contradicts the hypothesis on b. Hence π is a maximal σ-play so it is p-winning, and so is $m\pi$. Therefore σ' is winning from a, which proves that a is p-winning. □

Lemma 3. *Let p be a player. A p-position a is $\neg p$-winning if and only of if all p-moves from a lead to $\neg p$-winning positions.*

Proof. Suppose a is $\neg p$-winning and let σ be a winning $\neg p$-strategy. Consider a move $a \xrightarrow{m} b$. For each maximal σ-play $b \xrightarrow{\pi}$, $m\pi$ is a maximal σ-play from a so it is $\neg p$-winning, hence the play π is winning for $\neg p$. Therefore σ is winning from b so b is $\neg p$-winning.

Reciprocally, let a be a p-position. Consider the set B of target positions of moves from a. By hypothesis, for each $b \in B$ there exists a $\neg p$-strategy σ_b that is winning from b. Assume a well-ordering over B (this is not a constraint since in our setting B is always countable) and define the $\neg p$-strategy σ such that for each position x, $\sigma(x)$ is $\sigma_b(x)$ for the smallest b such that σ_b is winning from x, if such a b exists; $\sigma(x)$ is undefined otherwise.

Consider a maximal σ-play $\pi = a_0 \xrightarrow{m_0} a_1 \xrightarrow{m_1} a_2 \xrightarrow{m_2} \cdots$ with $a = a_0$. By recurrence we build a decreasing sequence $(b_i)_{i\geq 1}$ such that for each i, σ_{b_i} is winning from a_i and $\sigma(a_i) = \sigma_{b_i}(a_i)$ if a_i is a $\neg p$-move. Since $a_1 \in B$, by hypothesis σ_{a_1} is winning from a_1 and we take for b_1 the smallest b such that σ_b is winning from a_1. Now consider a move $a_i \xrightarrow{m_i} a_{i+1}$ and assume σ_{b_i} is winning from a_i. By construction, σ_{b_i} is winning from a_{i+1} too. If a_i is a $\neg p$-move then we can take $b_{i+1} = b_i$. If a_i is a p-move then we take for b_{i+1} the smallest b such that σ_b is winning from a_{i+1}, which implies $b_{i+1} \leq b_i$ by the remarks above.

Since the sequence (b_i) is decreasing for a well-founded order, it must be eventually constant with some value $b \in B$, hence π decomposes as $\pi_1\pi_2$ where π_1 is a σ-play and π_2 is a maximal σ_b-play. Therefore π_2 is $\neg p$-winning and so is π. This proves that σ is winning from a, hence a is $\neg p$-winning. $\square$

Theorem 1 (determinacy)**.** *Each position is winning for either* **P** *or* **O***.*

Proof. Consider the set S of positions that are not **P**-winning. By lemma 2, for each **P**-position a in S, for each move $a \xrightarrow{m} b$, the **O**-position b is in S. By lemma 3 for each **O**-position a in S, there exists a move $a \xrightarrow{m} b$ such that the **P**-position b is in S. Choosing one such move for each **O**-position in S, we get a **O**-strategy σ such that each σ-play starting from a position in S stays in S. Moreover, σ is defined for every position in S, so a maximal σ-play starting from a position in S cannot end on a **O**-position. Therefore all maximal σ-plays are **O**-winning. As a consequence, a position that is not **P** winning must be **O**-winning. $\square$

2. LK system

2.1. The Proof as a Mathematical Object

For centuries, mathematicians convinced each other of the truthfulness of their statements without delving too deeply into what was permissible and what was not in this process of persuasion. While it remains true that many proofs made in the past would not be considered sufficiently formal today, many mathematicians were able to correctly capture what would still be considered the key idea of the proof.

In this first section[1], we briefly discuss what led to the necessity of formalizing the concept of proof in mathematics. Before the end of the 1800s, except for sporadic attempts[2], there was never a concern for the formalization of the concept of a proof.

So, while it is reasonable to assume that every mathematician had an implicit idea of what a proof was, there was no perceived need to specify the idea of a proof as a mathematical object. An idea that was certainly widespread—at least since Greek philosophy—was that of correct reasoning as a *sequence of statements, each linked to the previous ones by logical rules*: consider that Aristotelian logic is based on syllogism, *i.e.*, a finite succession of inferences.

The turning point occurred with the pioneering works of several mathematicians, among whom G. Frege, D. Hilbert, and B. Russell are notable. Frege, in his *Begriffsschrift* (Frege 1879), advocated the logicist position according to which all of mathematics can ultimately be reduced to logic. This was where, for the first time, quantifiers were formally introduced, and consequently, the concept of quantified variables[3].

Subsequently, Hilbert—in his program (Hilbert 1920)—proposed to the mathematical community to demonstrate that the structure of mathematics was solid, that is, non-contradictory[4]. Hilbert envisioned the future of mathematical science as a discipline capable of providing indisputable truths, where new principles could be introduced and their compatibility with existing principles could be proved within the theory itself, rather than being justified through subjective opinions in generic discussions. Hilbert's focus was not on the formalization of the proof object per se, but rather on making mathematical procedures secure through controllable methods, which he referred to as *finitistic*. This intention, however, automatically brings with it

[1] This first section was written based on an interview with Professor Vito Michele Abrusci.

[2] For example, in the 16th century Maurolico attempted to rewrite Euclidean geometry using syllogisms.

[3] We are referring to quantifiers with a specific logical-mathematical meaning. The importance of quantifiers for the study of logic had already been recognized at least since Aristotle.

[4] A theory is contradictory if it proves a statement and its negation.

the need to formalize, in addition to the basic axioms, the concept of proof[5]; in light of axioms and logical rules of deduction, it is then hoped that mathematics will turn out to be non-contradictory.

Hilbert was not only one of the first to seek an adequate formalization for the concept of proof, but also one of the first to attempt to prove the non-contradiction of a theory, particularly notable in his attempt for analysis. Moreover, Hilbert is probably one of the first to realize that the intuitive idea of proof as a sequence of logical rules needs to be expanded with a **tree structure** (Abrusci 1985), where the root of the tree is the statement intended to be proven. Indeed, if the goal is to highlight which logical steps each formula depends on, a tree is a more suitable structure. In such a structure, a consequence may depend on multiple premises, and the same premise can be used multiple times.

However, the rules of derivation presented by Hilbert sparked in some mathematicians the need for a more natural and intuitive approach to logic. For instance, J. L. Łukasiewicz expressed this necessity in some seminars in 1926, an approach later embraced by S. Jaśkowski and G. Gentzen.

2.1.1. Natural Deduction

Gentzen's work was particularly crucial for the development of the entire theory of proof in the 1900s. He introduced **Natural Deduction**, which, while aiming to clarify and formalize what constitutes a proof, also seeks—as the name suggests—to focus on the natural rules of human reasoning. In other words, the goal is to find valid and sufficient rules of deduction that try to be as close as possible to the dynamics employed by a mathematician during their work.

> *My starting point was this: The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. Considerable formal advantages are achieved in return.*
>
> *In contrast, I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a calculus of natural deduction (NJ for intuitionist, NK for classical predicate logic). This calculus then turned out to have certain special properties; in particular the law of the excluded middle, which the intuitionists reject, occupies a special position. (Gentzen 1969)*

So, while Gentzen acknowledges the advantages of formalization seen by Hilbert, Frege, and Russell, he believes that these benefits can be achieved without straying too far from the dynamics of a mathematician's reasoning.

[5] By "formalize", we mean the act of *giving form within a mathematical context*, clearly outlining language, rules, and procedures.

In the following, we highlight the main features of natural deduction. Interested readers can find a more detailed discussion, both historical and mathematical, in (Veredice 2023b).

First and foremost, Gentzen clarified the language, that is, the set of all and only the symbols that can be used, specifying connectives ($\perp$, $\wedge$, $\vee$, $\rightarrow$[6]) and quantifiers ($\forall$, $\exists$). On this occasion, he introduced for the first time the symbol $\neg$ for the negation and $\forall$ for the universal quantifier (Gentzen 1935, p. 178), replacing Peano's previous symbol (). Peano had also introduced, among other things, the symbol $\exists$ for the existential quantifier [7] (Peano, Vailati, Pieri, et al. 1896, p. IX).

Subsequently, Getnzen provides deduction rules (Figure 2.1), which allow moving from one set of formulas to another, distinguishing between introduction rules (I) and elimination rules (E) for each logical symbol. For each rule, the formulas above the line are called the *premises* of the rule, and the formula below the line is the *conclusion*. The acronym to the right of the line indicates the name of the rule.

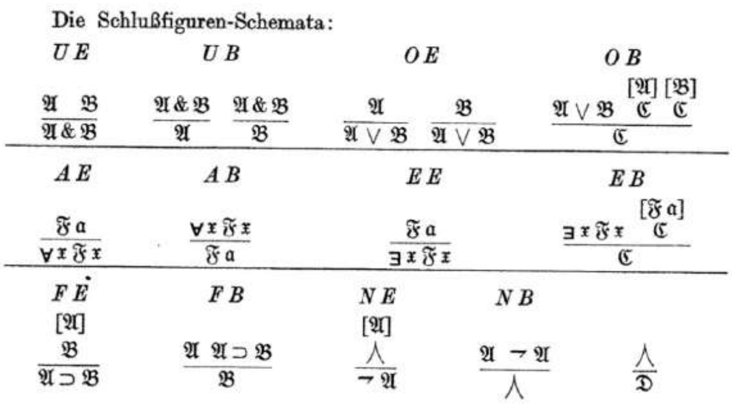


Figure 2.1.: Extract from an original work of Gentzen (1935), where deduction rules are presented.

Let's now present an updated version of Natural Deduction, with modernized symbolism.

These are the three introduction and elimination rules for the logical connective $\wedge$.

$$\frac{A\wedge B}{A}\wedge E \qquad \frac{A\wedge B}{B}\wedge E \qquad \frac{A \quad B}{A\wedge B}\wedge I$$

In other words, if one knows $A\wedge B$, one can deduce both A and B (elimination rules), and if both are known, one can deduce their conjunction (introduction rule).

The following rules are for the logical connective $\vee$.

[6]In Gentzen's original work, implication and the $\perp$ symbol were denoted with different symbols.

[7]It is interesting to note that Gentzen attributes the $\exists$ symbol to Russell, since Russell had made extensive use of Peano's symbolism. Furthermore, the symbolism introduced by Gentzen did not become standard in mathematical practice until the 60s-70s. For instance, Tarski (1959) utilized different symbols for quantifiers (*i.e.*, $\bigwedge$ and $\bigvee$). Moreover, Miro Quesada (1968)[p.211] presents the symbol $\exists$ (though erroneously attributing it to Russell) and the symbol () for the universal quantifier.

$$\frac{A}{A \vee B} \vee I \qquad \frac{B}{A \vee B} \vee I \qquad \frac{A \vee B \quad \begin{array}{c}[A] \\ \vdots \\ C\end{array} \quad \begin{array}{c}[B] \\ \vdots \\ C\end{array}}{C} \vee E$$

As far as the introduction rules are concerned, they simply tell the trivial fact that if you can prove something then you can prove that something $\vee$ something else. Regarding the elimination of the connective $\vee$, the rule indicates that if one knows $A \vee B$ and knows as well that both A and B lead to the conclusion C after a certain number of steps, then one can deduce C. Notably, in both the introduction of $\wedge$ and the elimination of $\vee$, the utility of a tree structure is shown.

The following rules regard the implication $\rightarrow$.

$$\frac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \rightarrow B} \rightarrow I \qquad \frac{A \quad A \rightarrow B}{B} \rightarrow E$$

We underline that (→E) is usually called Modus Ponens.

The system of rules $\wedge I$, $\wedge E$, $\vee I$, $\vee E$, $\rightarrow I$, $\rightarrow E$ identifies what is known as minimal logic. If the following rule ($\perp E$) is added to these, which embodied the famous expression "ex falso sequitur quodlibet", we obtain a system of rules for intuitionistic logic.

$$\frac{\perp}{A} \perp E$$

Within classical logic, the principle of the excluded middle, $A \vee \neg A$, also holds, where $\neg A = A \rightarrow \perp$. Therefore, the system of rules for classical logic includes the already discussed rules of inference for intuitionistic logic, along with the principle of *Reductio ad absurdum*:

$$\frac{\begin{array}{c}[\neg A] \\ \vdots \\ \perp\end{array}}{A}$$

To expand propositional logic to first-order logic, it is necessary to add rules for each quantifier.

$$\frac{\begin{array}{c}\Gamma \\ \vdots \\ A(x)\end{array}}{\forall x A(x)} \forall I \text{ where the variable } x \text{ is not free in } \Gamma \qquad \frac{\forall x A(x)}{A\left(\frac{t}{x}\right)} \forall E \qquad \frac{A\,(t/x)}{\exists x A(x)} \exists I$$

$$\dfrac{\exists x A(x) \qquad \begin{matrix} & [A(x)] \\ \Gamma\ddots & \vdots \\ & C \end{matrix}}{C}\ \exists E \text{ where the variable } x \text{ is not free in } C \text{ nor in } \Gamma.$$

In other words,

- if $A(x)$ is true for an x about which no particular assumptions have been made, then $\forall x A(x)$ is also true (introduction of 'for all')
- If $\forall x A(x)$ is true, then $A(t)$ is also true, where t is a term of choice (elimination of 'for all')
- If $A(t)$ is true for some term t, then there exists an x for which $A(x)$ is true (introduction of 'exists')
- If one knows that $\exists x A(x)$ is true and something follows from $A(x)$, that something can be deduced (elimination of 'exists')

As we note, sometimes formulas appear in square brackets. A formula within square brackets is a *discharged hypothesis*. This means that the result holds true even without assuming that hypothesis. Clearly, if in a tree every leaf (*i.e.*, every hypothesis) has been discharged, then a *tautology* has been proven, which is a statement that is always true.

For clarity, we provide the reader with a very simple example of derivation in natural deduction: from A, $A \to B$ and $B \to C$, one can deduce C.

$$\dfrac{\dfrac{A \qquad A \to B}{B}\ \to \mathrm{E} \qquad B \to C}{C}\ \to \mathrm{E}$$

2.1.2. LK sytem

Returning to Hilbert's intention to demonstrate the non-contradiction of mathematics, an extremely clever strategy for proving the non-contradiction of a theory is that of the *purity of methods*.

More specifically, if one can show that in a certain theory, to prove a particular statement F, it is possible to avoid referring to something external to F, then the theory is certainly non-contradictory. This is because any contradictory theory proves the false—*i.e.*, $\perp$—and since $\perp$ has no subformulas, it certainly cannot be proven within a theory that enjoys the purity of methods[8], that is, the property commonly known as *cut elimination*. The name is emblematic and clarifying: the cut is the logical rule that allows one to "cut", that is, to add something impure and extraneous to a

[8] Of course, one assumes that false is not an axiom.

proof. If it is possible to eliminate the rule of the cut[9] without losing any theorem, then the theory in question enjoys the property of cut elimination.

Gentzen thus faced the challenge of *cut elimination.* His basic idea for proving cut elimination was to treat proofs as mathematical objects and define transformations on these. The idea was that these transformations would be a way to obtain a cut-free proof from a proof with cuts. However, Gentzen soon realized that natural deduction was too rudimentary a tool for deduction and insufficient for his purposes. Therefore, the system evolved, leading to the famous formulation of the *LK system* (der Logistische Kalkül) in 1935 (Gentzen 1935).

The LK logical system is centered around the concept of a *sequent*; indeed the system is also known as the sequent calculus. A sequent is an implication between two multisets[10] of formulas, commonly expressed as $\Gamma \vdash \Delta$, where Γ and Δ are sets of formulas. More precisely, the notation $\Gamma \vdash \Delta$ says that from the conjunction of all formulas in Γ, one can deduce the disjunction of the formulas in Δ. The LK system consists of a set of rules for deducing one sequent from others. It is interesting to underline the shift from Natural Deduction, where from certain information new information can be deduced, in LK, where from the fact that from certain information others are deduced, one deduces that from other information yet more are deduced. One could argue that this reflects a more structured approach to the process of deduction, emphasizing the relational aspects between sets of statements.

In the LK system, there are three kinds of rules: **identity rules**, **structural rules**, and **logical rules**. The logical rules work on the logical structure of a particular formula, that is, on its connectives and quantifiers, just as the rules of Natural Deduction do. Structural rules, on the other hand, work on the structure of the proof itself. The structural rules include weakening (adding a formula to one side of the sequent) and contraction (replacing multiple occurrences of a formula with a single occurrence). The logical rules consist of introduction and elimination rules for each logical connective and quantifier, much like in Natural Deduction.

Identity Rules

$$\frac{}{A \vdash A}\ (\mathrm{ax}) \qquad\qquad \frac{\Gamma \vdash \Delta, A \qquad A, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}\ (\mathrm{cut})$$

Logical Rules

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge L_1 \qquad\qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee R_1$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge L_2 \qquad\qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee R_2$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Sigma, B \vdash \Pi}{\Gamma, \Sigma, A \vee B \vdash \Delta, \Pi}\ \vee L \qquad\qquad \frac{\Gamma \vdash A, \Delta \qquad \Sigma \vdash B, \Pi}{\Gamma, \Sigma \vdash A \wedge B, \Delta, \Pi}\ \wedge R$$

[9] *I.e.*, not allowing the possibility of performing the operation.
[10] Intuitively, a multiset is a set in which an element can appear multiple times.

$$\frac{\Gamma \vdash A, \Delta \qquad \Sigma, B \vdash \Pi}{\Gamma, \Sigma, A \to B \vdash \Delta, \Pi} \to L \qquad\qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \to R$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \neg L \qquad\qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \neg R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta} \forall L \qquad \frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \forall x A, \Delta} \forall R \text{ (where } y \text{ is not free in } \Gamma \cup \Delta)$$

$$\frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta} \exists L \text{ (where } y \text{ is not free in } \Gamma \cup \Delta) \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta} \exists R$$

Structural Rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{WL} \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{WR}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{CL} \qquad\qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{CR}$$

The reader can interpret these rules following the guidelines provided in the section on Natural Deduction. Let us anyway briefly discuss the Identity Rules and the Structural Rules.

There are two identity rules: (ax) and (cut). The cut, understood as the addition of something impure, should be interpreted from bottom to top: if one want to derive the sequent $\Gamma, \Sigma \vdash \Delta, \Lambda$, the proof can be cut with any formula A deemed suitable for the purpose. On the other hand, (ax) asserts that $A \to A$ is always derivable, for every formula A.

As for the structural rules, we have W (Weakening) and C (Contraction). Weakening allows the addition of material to the left side or right side; this does not change the truth value of what is being proven if we remember that the comma to the left of $\vdash$ should be read as $\wedge$ and the comma to the right of $\vdash$ as $\vee$. Contraction implies that if something can be done with multiple copies of A, it can also be done with a single A. Two key aspects are worth highlighting: firstly, it is intriguing to interpret contraction as *consequentia mirabilis*, moving one of the two As from the premise to the other side of $\vdash$. Secondly, the structural rules are a typical element of distinction between different logical systems; in particular, linear logic aims to thoroughly clarify the role of structural rules.

Finally, we emphasize that when working in mathematics, one does not operate in a "pure" environment as just described, but within the context of a theory **T**, which specifies the axioms, *i.e.*, those formulas that can be assumed as true in one's derivations.

To work with a theory **T**, the following rule (axT) is therefore added, which states that every formula F of the theory is always derivable.

$$\frac{}{\vdash F} \text{(axT) with } F \in \mathbf{T}$$

The LK logical system is fundamental in today's logic for several reasons. Firstly, it provides a powerful and flexible proof system, capable of expressing complex reasoning patterns. Additionally, the LK system and its sub-systems serve as a foundation for computational logic, playing a crucial role in formal verification, automated reasoning, and the semantics of programming languages.

The notion of cut-elimination in the LK system is profound, with its significance extending beyond the purposes for which it was initially conceived. This process can be seen as a kind of "program execution", where cut-elimination corresponds to the interaction and unfolding of computational procedures. In this sense, Gentzen's approach to cut-elimination has been extremely influential in the development of numerous modern branches of logic. Broadening the context, we can mention the Curry-Howard correspondence between computer programs and mathematical proofs, where eliminating the cut equates to executing the program.

2.1.3. General Observations

Despite the unattainable nature of Hilbert's dream, the quest to fulfill it has catalyzed an abundance of research and discoveries. We might indeed recognize in foundational crisis[11] and the subsequent works of many authors, among whom we remember R. Carnap, A. Church, H. Curry, K. Gödel, E. L. Post, A. Tarski, A. Turing, and E. Zermelo, one of the most fertile periods of logic.

Numerous studies have followed on the nature of proofs and the most appropriate ways to represent them. Among various developments, it is essential to mention Linear Logic (Girard 1987) and Proof Nets (Girard 1989). Linear logic, in particular, offers a new perspective on resource management in the proof process and a deeper understanding of connectives, while proof nets provide a graphical representation of proofs, highlighting their structural and interactive properties. These advancements demonstrate the ongoing evolution and significance of proof theory in the fields of logic, mathematics, and computer science.

While it remains true, today as in the past, that a mathematician—in their daily practice—is usually not interested in discussing the proof as a mathematical object, it is undeniable that the pursuit of rigor has markedly intensified, influencing both academic research and education.

Furthermore, anyone dedicated to exploring the nature of mathematical proof as both a social construct and an educational tool must acquaint themselves with the efforts to formalize proof as a distinct mathematical object.

2.2. From LK to LK$_{game}$

To begin with, it is noteworthy that, although the standard language includes all connectives and quantifiers and the rules are defined for each quantifier and con-

[11] The *foundational crisis* of mathematics occurs precisely when the mathematical community is seriously concerned that mathematics does not rest on solid foundations.

nective, "the same results" are attainable with simpler systems, made with fewer quantifiers and connectives. For example, the conjunction $\wedge$ can be obtained using the disjunction $\vee$ and the negation.

We present here the fragment $LK_{\forall\to}$ of classical logic that is Gentzen's LK restricted to the rules for the connective $\to$ and the quantifier $\forall$. In the following, let $\mathbf{T}$ be a first order theory.

Identity Rules

$$\frac{}{\Gamma, \mathbf{A} \vdash \mathbf{A}, \Delta}\ (\text{ax}) \qquad \frac{}{\Gamma \vdash \mathbf{F}, \Delta}\ (\text{axT}) \text{ with } F \in \mathbf{T} \qquad \frac{\Gamma, A \vdash \Delta \qquad \Sigma \vdash A, \Lambda}{\Gamma, \Sigma \vdash \Delta, \Lambda}\ (\text{cut})$$

Structural Rules

$$\frac{A, A, \Gamma \vdash \Delta}{\mathbf{A}, \Gamma \vdash \Delta}\ (\text{CL}) \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, \mathbf{A}}\ (\text{CR})$$

Logical Rules

$$\frac{A(t/x), \Gamma \vdash \Delta}{\forall \mathbf{x} \mathbf{A}(\mathbf{x}), \Gamma \vdash \Delta}\ (\forall\text{L}) \qquad \frac{\Gamma \vdash \Delta, A(y/x)}{\Gamma \vdash \Delta, \forall \boldsymbol{x} \boldsymbol{A}(\boldsymbol{x})}\ (\forall\text{R}) \text{ with } y \text{ fresh variable}$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Sigma \vdash \Lambda}{\mathbf{A} \to \mathbf{B}, \Gamma, \Sigma \vdash \Delta, \Lambda}\ (\to\text{L}) \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, \mathbf{A} \to \mathbf{B}}\ (\to\text{R})$$

We recall that sequents are multiset of formulas. In the following, referring to a formula, we will refer to a particular occurrence in the sequent of that formula. Moreover:

- In the sequent conclusion of a rule, bold formulas are said to be *main* in the rule;
- In the sequent premises of a rule, red formulas are said to be *active* in the rule;
- in the sequent $\Gamma \vdash \Delta$ any formula in Γ is called left formula and any formula in Δ is called right formula;
- A logical rule is said to be irreversible if its main conclusion is a left formula and reversible if its main conclusion is a right formula.

It is well known, see for example (Troelstra and Schwichtenberg 2000), that one can restricts to axioms having atomic formulas as main formulas, and by "pushing weakening rules towards the axioms" any proof of Gentzen LK can be transformed to a proof with axioms of the shape considered above and without weakening.

We also note that there are no rules for the $\perp$ connective. As a result, the system is not equivalent to standard LK . However, at the end of the chapter, we will show how to achieve equivalence with classical logic by adding some requirements.

2.2.1. Cut-elimination

The purpose of this section is to proof cut-elimination for this particular fragment of LK. In other words, we will show that the cut rule, which allows the possibility to introduce new formulas extraneous to the sequent being proved, is only necessary when cutting with a formula of the theory.

To do so, let us consider an occurrence of the cut rule. The general form of a cut rule in a derivation π is as follows, where R1 and R2 are arbitrary rules:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, A \vdash \Delta}\,\text{R1} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Sigma \vdash A, \Lambda}\,\text{R2}}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,\text{(cut)}$$

To demonstrate cut-elimination, as discussed in the previous section, we will define transformations on the proof π. Since the transformations we define will only modify the rules above the cut rule, we can assume without loss of generality that the (cut) rule shown is the last rule of the derivation of π. Furthermore, all our derivations will preserve the sequent conclusion $\Gamma, \Sigma \vdash \Delta, \Lambda$.

We now define the set of transformations $\mathscr{T}$, based on the types of rules R1 and R2 in π. To start with, we consider the simplest case where rules R1 and R2 have as their principal formula the active formula A in the cut. If R1 and R2 are logical rules, we have two possible cases: R1 and R2 are respectively $(\rightarrow L)$ and $(\rightarrow R)$ or $(\forall L)$ and $(\forall R)$.

(→) case

$$\dfrac{\dfrac{\begin{matrix}\pi_1'\\ \vdots\\ \Gamma \vdash \Delta, B\end{matrix} \qquad \begin{matrix}\pi_1''\\ \vdots\\ C, \Gamma' \vdash \Delta'\end{matrix}}{\Gamma, \Gamma', B \rightarrow C \vdash \Delta, \Delta'}\,(\rightarrow\text{L}) \qquad \dfrac{\begin{matrix}\pi_2'\\ \vdots\\ B, \Sigma \vdash \Lambda, C\end{matrix}}{\Sigma \vdash B \rightarrow C, \Lambda}\,(\rightarrow\text{R})}{\Gamma, \Gamma', \Sigma \vdash \Delta, \Delta', \Lambda}\,\text{(cut)}$$

In this case, we define the transformation that transforms π into the following derivation.

$$\dfrac{\dfrac{\begin{matrix}\pi_2'\\ \vdots\\ B, \Sigma \vdash \Lambda, C\end{matrix} \qquad \begin{matrix}\pi_1'\\ \vdots\\ \Gamma \vdash \Delta, B\end{matrix}}{\Gamma, \Sigma \vdash \Delta, \Lambda, C}\,\text{(cut)} \qquad \begin{matrix}\pi_1''\\ \vdots\\ C, \Gamma' \vdash \Delta'\end{matrix}}{\Sigma, \Gamma, \Gamma' \vdash \Delta, \Delta', \Lambda}\,\text{(cut)}$$

The sequent conclusion is the same as in π, and no structural rules have been added. As one may notice, we transformed the cut into two cuts. However, both cuts are applied to subformulas of $B \rightarrow C$. Intuitively, each of the two new cuts has an active formula of strictly lesser complexity than the active formula in the initial cut: this aspect is fundamental for the induction that we will later define. We note that there is an arbitrariness in the procedure, because one can choose which cut to execute first (whether on B or on C).

(∀) case

$$\dfrac{\dfrac{\begin{matrix}\pi_1' \\ \vdots \\ \Gamma, C[t/x] \vdash \Delta\end{matrix}}{\Gamma, \forall x C(x) \vdash \Delta}\,(\forall L) \qquad \dfrac{\begin{matrix}\pi_2' \\ \vdots \\ \Sigma \vdash \Lambda, C[y/x]\end{matrix}}{\Sigma \vdash \forall x C(x), \Lambda}\,(\forall R)}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,(\text{cut})$$

The basic idea is simple: we commute the cut with the respective quantifier introduction rules: in this way, we transform the cut into another cut where the active formula has a lower complexity. However, the details are complicated, as careful handling of variables is required. For a complete treatment, refer to (Abrusci and Tortora de Falco 2014).

Let us now consider the case where it's not true that both R1 and R2 are logical rules.

(ax)

We note that a formula can be introduced by a rule (ax) either as the main formula or within the context. If at least one of the two derivations (for example, π_1) consists of a single rule which is an axiom conclusion rule $\Gamma, A \vdash A, \Delta$, where A is the active formula in the cut rule, then we simply consider only the other derivation π_2, weakening one of its axioms with Γ on the left and Δ on the right. Even in this case, there is an arbitrariness if both rules (R1) and (R2) are axioms with A as main formula. Moreover, this procedure is not deterministic because we need to choose on which axiom of π_2 we will do the weakening.

The second case is where the main formula of the axiom is not active in the cut. Traditionally, this would involve working on weakening, which we have incorporated into the axiom. In this case, the opposite of what was done previously is carried out: we weaken on π_1 with Σ and Λ, and do not weaken on A.

(axT)

If an axiom F is the active formula in the cut, thus R2 = (axT), the cut cannot be eliminated. However, we can work on the structure of the derivation to obtain a specific cut rule for the axioms of the theory, which will be the only one remaining in our new system.

$$\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ \Gamma, F \vdash \Delta\end{matrix} \qquad \dfrac{}{\Sigma \vdash F, \Lambda}\,(\text{axT})}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,(\text{cut}) \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\pi_1 \\ \vdots \\ \Gamma, F \vdash \Delta\end{matrix}}{\Gamma \vdash \Delta}\,(\text{if } F \in T)$$

In this cut, we weaken on π_1 with the contexts Σ and Λ. The idea is that, by looking at the derivation from the bottom up, we can always assume the presence of axioms in the left context.

If F is not active in the cut, then the situation is analogous to the handling of the axiom (ax), because it is always a weakening.

(C)

Let's now assume that at least one of the two rules R1 and R2 is a contraction with the main conclusion being the active formula in the cut rule. Without loss of generality, suppose that R1 is a contraction with A as principal conclusion.

$$\dfrac{\begin{matrix}\pi_1' \\ \vdots \\ \Gamma, A, A \vdash \Delta\end{matrix}}{\Gamma, A \vdash \Delta}\ (\mathrm{CL})$$

In this case, the transformation is as follows.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1' \\ \vdots \\ \Gamma, A, A \vdash \Delta\end{matrix} \qquad \begin{matrix}\pi_2' \\ \vdots \\ \Sigma \vdash A, \Lambda\end{matrix}}{\Gamma, \Sigma, A \vdash \Lambda, \Delta}\ (\mathrm{cut}) \qquad \begin{matrix}\pi_2' \\ \vdots \\ \Sigma \vdash A, \Lambda\end{matrix}}{\Gamma, \Sigma, \Sigma \vdash \Lambda, \Lambda, \Delta}\ (\mathrm{cut})}{\Gamma, \Sigma \vdash \Lambda, \Delta}\ \mathrm{CL/CR}$$

This time—as can be seen—we have added (many) structural rules, and the two new cuts do not have active formulas of lesser complexity. It is important to note that if both are contractions, we must make a choice on which to duplicate. This is the case that complicates the entire procedure and shows how a cut-free derivation transformed from a derivation with cuts can be significantly longer and more complex.

(Commutative Cut)

Now let's analyze the last remaining case, where at least one of the two rules R1 and R2 does not have as its main conclusion the active formula in the cut rule. The idea is that in this case a permutation of rules is performed to "seek out" the rule that introduces the active formula in the cut rule. Suppose, for instance, that (R1) is a rule with the main conclusion being the occurrence of a formula $C \in \Gamma$.

If R1 is zero-ary, then R1 is either (ax) or (axT). These cases have already been dealt with above as weakening.

If R1 is unary, then regardless of the rule, we will have the following situation.

$$\dfrac{\dfrac{\begin{matrix}\pi_1' \\ \vdots \end{matrix}}{A, \Gamma' \vdash \Delta'}\ \mathrm{R1'}}{A, \Gamma \vdash \Delta}\ \mathrm{R1}$$

In this case, the transformation is the following.

$$
\dfrac{\dfrac{\dfrac{\pi_1'}{\vdots}}{A, \Gamma' \vdash \Delta'}\,\mathrm{R1}' \qquad \dfrac{\dfrac{\pi_2}{\vdots}}{\Sigma \vdash \Lambda, A}\,\mathrm{R2}}{\dfrac{\Gamma', \Sigma \vdash \Delta', \Lambda}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,\mathrm{R1}}\,\mathrm{cut}
$$

In other words, we commute the (cut) with R1.

If R1 is binary, it could be (cut) or (→L). If R1 is (→L) and that A appears in the left premises.

$$
\dfrac{\dfrac{\pi_1'}{\vdots} \quad \dfrac{\pi_1''}{\vdots}}{} \qquad \dfrac{\Gamma', A \vdash \Delta', B \qquad C, \Gamma'' \vdash \Delta''}{\Gamma, A \vdash \Delta}\,(\rightarrow\mathrm{L})
$$

The transformation is the following.

$$
\dfrac{\dfrac{\dfrac{\pi_1'}{\vdots}\quad \Gamma', A \vdash \Delta', B \qquad \dfrac{\dfrac{\pi_2}{\vdots}}{\Sigma \vdash \Lambda, A}\,(\mathrm{R2})}{\Gamma', \Sigma \vdash \Lambda, \Delta', B}\,\mathrm{cut} \qquad \dfrac{\pi_1''}{\vdots}\quad C, \Gamma'' \vdash \Delta''}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,(\rightarrow\mathrm{L})
$$

On the other hand, if R1 is a (cut) we have the following situation.

$$
\dfrac{\dfrac{\pi_1'}{\vdots}\quad \Gamma', A, B \vdash \Delta' \qquad \dfrac{\pi_1''}{\vdots}\quad \Gamma'' \vdash B, \Delta''}{\Gamma, A \vdash \Delta}\,(\mathrm{cut})
$$

In which we transform as follows.

$$
\dfrac{\dfrac{\dfrac{\pi_1'}{\vdots}\quad \Gamma', A, B \vdash \Delta' \qquad \dfrac{\dfrac{\pi_2}{\vdots}}{\Sigma \vdash \Lambda, A}\,(\mathrm{R2})}{\Gamma', \Sigma, B \vdash \Lambda, \Delta'}\,\mathrm{cut} \qquad \dfrac{\pi_1''}{\vdots}\quad \Gamma'' \vdash B, \Delta''}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,(\mathrm{cut})
$$

So, in essence, on an intuitive level, regardless of the rule—whether zero-ary, unary, or binary—the cut being eliminated can be progressively moved upwards until the active formulas in the cut rule are both principal.

2.2.1.1. $\mathscr{T}_{glob}$

It is important to note that if we apply—for example—the transformation of the CC (Commutative Cut) to swap two cuts and then reapply the transformation to the other cut, what results is an infinite process where the two cuts are continuously swapped. In other words, if the transformations of $\mathscr{T}$ just presented are not applied judiciously, one can end up with a procedure that does not terminate.

For this reason, we define $\mathscr{T}_{glob}$, which tells how these transformations can be composed to ultimately result in a procedure that terminates in a cut-free derivation.

Cuts can be divided into two categories: logical cuts and structural cuts, *i.e.*, those that do not operate on the logical structure of the formula. The logical cuts (L) are the two cuts shown at the beginning of the section, named ($\rightarrow$) and ($\forall$). All other cuts are referred to as elementary structural steps.

$\mathscr{T}_{glob}$ is a set of transformations that contains the logical steps L as well as the structural steps S, which are compositions of elementary structural steps. Therefore, we define the generic structural step S, that is, when at least one of the two rules R1 and R2 is not a logical rule that introduces the active formula in the cut, and thus a logical step cannot be applied.

More precisely, if R1 is not a logical rule where A is principal, we trace the *history* of A, that is, where A was introduced. The history of A is a tree because A may have been introduced on various different occasions and then contracted. In particular, A could have been introduced in the following ways.

$$\dfrac{\dfrac{}{\Gamma_1, A \vdash \Delta_1}\,(\text{Log}) \quad \dfrac{}{\Gamma_2, A \vdash A, \Delta_2}\,(\text{ax}) \quad \dfrac{}{\Gamma_3, A, B \vdash B, \Delta_3}\,(\text{ax}) \quad \dfrac{}{\Gamma_4, A \vdash F, \Delta_4}\,(\text{axT})}{\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots}$$
$$\dfrac{\qquad\qquad\qquad\qquad\qquad\qquad}{\Gamma, A \vdash \Delta}\,(\text{R1})$$

Clearly, if A was introduced as the main formula of an axiom (ax), being atomic, it cannot have also been introduced by a logical rule (Log). However, this tree demonstrates all the possible cases of introduction.

Let us then define the following structural step, which we will call S_1.

- Where A is introduced with a logical rule, commute the cut until reaching the logical rule.
- If A is introduced as the main formula of the axiom, replace the axiom rule with the derivation π_2, where one of the axioms is weakened with Γ_2 on the left and Δ_2 on the right.
- If A is introduced as context of the axiom, then do not weaken on A but on Σ and Λ.
- If A is introduced as context of (axT), then do not weaken on A but on Σ and Λ.

Let us now suppose that R1 is a logical rule that introduces A. If R2 is also a logical rule introducing A, then we are dealing with a logical cut L.

On the other hand, if R2 is not a logical rule introducing A we are dealing with a structural cut that we will call S_2. We trace the history of A, *i.e.*, the set of rules that have introduced A. As already discussed, this history is a tree.

$$\dfrac{\dfrac{}{\Sigma_1 \vdash A, \Lambda_1}\,(\text{Log}) \quad \dfrac{}{\Sigma_2, A \vdash A, \Lambda_2}\,(\text{ax}) \quad \dfrac{}{\Sigma_3, B \vdash B, A, \Lambda_3}\,(\text{ax}) \quad \dfrac{}{\Sigma_5 \vdash F, A, \Lambda_4}\,(\text{axT}) \quad \dfrac{}{\Sigma_4 \vdash A, \Lambda_5}\,(\text{axT})}{\Sigma \vdash A, \Lambda}\,(\text{R2})$$

The definition of S_2 is entirely analogous and symmetric to that of S_1, with the only difference being that if A is introduced as an axiom by (axT), then the cut cannot be eliminated. The transformation will be limited to commuting the cut up to the (axT) rule, and at that point, the cut will be transformed into the form discussed previously. It's important to make a clarification that will be useful for what follows. Let A be an axiom introduced by the rule (axT), which will then become the active formula in the cut. If the left occurrence of A, active in the cut, never is—in its history—the main conclusion of a logical rule or an axiom, then—even in this case—the cut can be eliminated: it suffices not to weaken with that formula on the left.

Therefore, as already mentioned, a structural step is obtained by composing elementary structural steps presented previously.

Now, let's illustrate a procedure that, by applying the logical and structural steps just defined, allows for the elimination of the cut. It's important to note that the three types of cut are orderable: $S_1 > S_2 > L$.

In what follows, we will define a *quasi cut-free* derivation as a derivation that contains a single occurrence of the cut rule, which, as previously stated, can be considered to be the last. We will call $deg(\pi)$, where π is a quasi cut-free derivation, the complexity of the cut formula, *i.e.*, the total number of connectives and quantifiers. On the other hand, we will call the *energy* of the cut 0, 1, 2 if the cut is respectively L, S_2, S_1.

Lemma 4. *Given π as a quasi cut-free derivation, if the cut is of type S_1, then by applying a structural step, a derivation is obtained where all cuts are S_2. If it's of type S_2, then a derivation is obtained where all cuts are of type L. In other words, by applying a $\mathcal{T}_{glob}$ step, a derivation with possibly a greater number of cuts is obtained, but all of these cuts have a lower energy.*

Proof. The proof is evident by looking at how the structural steps S have been defined. □

Theorem 2 (Cut Elimination). *If π is a quasi cut-free derivation of $\Gamma \vdash \Delta$, then by applying transformations from the set $\mathcal{T}_{glob}$, π can be transformed into a derivation π' that does not contain cuts.*

Proof. The proof is carried out by induction on the pair (deg, energy), ordered lexicographically. If the cut in π is of type L, then applying a logical step yields a derivation with one or two cuts, both of a strictly lower degree.

Conversely, if the cut is of type S_2, then applying a structural step results in a derivation containing a number of cuts, each with the same active formula as in the S_2 cut, but all of type L. Thus, the degree does not increase.

Similarly, if the cut is of type S_1, then applying a structural step leads to a derivation containing several cuts, each with the same active formula as in the S_1 cut, but all of type S_2 or L. Therefore, in this case too, the degree does not increase. □

Having eliminated the cut, we can now define some concepts that will be useful in the following section.

Definition 11 (Anchestor and Residue)**.** *After a reduction step in $\mathcal{T}_{glob}$, the Ancestor of a logical rule or a cut of the theory is defined as the unique rule in the tree prior to the reduction from which this rule derives. Consequently, the Residue is defined as the counterimage of this function. Additionally, we specify that in the case of a logical cut, the two cuts created are not residues of the eliminated cut.*

2.2.2. Focusing

A standard distinction in proof-theory is between reversible and irreversible rules. More recently, also thanks to the work in Denotational Semantics (Amadio and Curien 1998) and linear logic (Girard 1991; Danos, Joinet, and Schellinx 1997; Laurent, Quatrini, and Tortora de Falco 2005) the focus shifted from rules to formulas. Intuitively, an irreversible (resp. reversible) formula is the main formula of an irreversible (resp. reversible) rule.

Definition 12 (Irreversible and reversible formulas)**.** *Let π be a derivation and F an occurrence of a non atomic formula in π. F is said to be irreversible if it is a left formula and reversible if it is a right formula.*

The aim of this section is to derive a new system LK_{game}that is more appropriate for interpreting $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ games. The new system will be equivalent to $LK_{\forall,\to}$ in the sense that it will prove the same set of formulas. We will begin by defining the order of multisets, which will be used frequently in the following discussion.

Definition 13 (Multiset Order)**.** *Let μ and ν be finite multisets mapping from $\mathbb{N}$ to itself. We define $\mu < \nu$ if there exists $k \in \mathbb{N}$ such that $\mu(k) < \nu(k)$ and for all $k' > k$, we have $\mu(k') = \nu(k')$.*

Definition 14 (Focused Proof)**.** *We say that a proof in $LK_{\forall\to}$ is* focused *if each left formula which is active in an irreversible rule or in a cut is the main conclusion of a logical rule or an axiom*[12].

[12] Since the main conclusions of axioms are atomic formulas only, this means that if the left formula is an irreversible formula then it is main in a logical rule.

The first question we ask ourselves is whether a focused proof is stable under cut-elimination. In other words, we want to understand if removing the cut from a focused derivation with cuts results in a focused derivation without cuts, or if it results in a cut-free derivation that is no longer focused. Clearly, our derivation system does not have the cut except in the special form of a cut with a formula of the theory, so the ultimate goal is to justify the interaction between two focused proofs without cuts.

Lemma 5. *Let* a *be a logical rule or a cut of the theory, and A an active left occurrence in* a*, such that this same occurrence is main in* m*, with* m *being an axiom or a logical rule.*

$$\cfrac{\cfrac{\vdots}{\Gamma, A \vdash \Delta}\,\mathrm{m}}{\vdots}\,\mathrm{a}$$

Then A is main in every residue of a.

Proof. Let c be the cut on which a transformation of $\mathcal{T}_{glob}$ is working, and let C be the active formula in c. If no ancestor of the formula C is in $\Gamma \cup \Delta \cup A$, then clearly the transformation does not modify either the structure or the sequence of rules m and a. Let us now assume that there exists an ancestor of C in $\Gamma \cup \Delta \cup \{A\}$, but the rule m does not introduce the active formula in the cut rule. In other words, suppose that $C \in \Gamma \cup \Delta$. During the reduction step of $\mathcal{T}_{glob}$, the rule a may be in the tree that rises or in the one that does not. If it is in the one that does not rise, then it is simply traversed by the CC and that block remains intact, and the active premise in a will continue to be principal in m. In this case, a has a single residue. On the other hand, if the sub-derivation is in the tree that rises, a will have several residues, but still, its structure will not be altered. If m, on the other hand, introduces the active formula in the cut, this must necessarily be A, which turns out to be an ancestor of the active formula C in the cut c on which $\mathcal{T}_{glob}$ is being executed.

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\Gamma, A \vdash \Delta}\,\mathrm{m}}{\vdots}\,\mathrm{a}}{\Gamma', A \vdash \Delta'}\,(\mathrm{R1}) \qquad \cfrac{}{\Sigma \vdash A, \Lambda}\,(\mathrm{R2})}{\Gamma', \Sigma \vdash \Delta', \Lambda}\,\mathrm{cut}$$

If the rule m introduces exclusively the formula A on the left[13], then necessarily the rule a is the cut on which we are working. Otherwise, A would no longer be present after being active in a. In this case, $\Gamma = \Gamma'$ and $\Delta = \Delta'$.

$$\cfrac{\cfrac{}{\Gamma, A \vdash \Delta}\,\mathrm{m} \qquad \cfrac{}{\Sigma \vdash A, \Lambda}\,(\mathrm{R2})}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,\mathrm{cut}$$

[13]This means m is not axiom.

If the cut is logical, a leaves no residue, and thus the fact is trivially proven. If the cut is of type S2, then π_1 is the ascending sub-derivation, which remains unchanged.

The last and most interesting case is where m is an axiom rule with $\Gamma, A \vdash A, \Delta$ as a conclusion. The left occurrence of A will then become immediately active in a, while the right occurrence of A will subsequently become active in the cut c.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A \vdash A, \Delta}\,(\mathrm{ax})}{}\,\mathrm{a}}{\dfrac{\vdots}{\Sigma \vdash A, \Lambda}\,\mathrm{R2}} \qquad \dfrac{\dfrac{\pi_1}{\vdots}}{\Gamma', A \vdash \Delta'}\,\mathrm{R1}}{\Gamma', \Sigma \vdash \Delta', \Lambda}\,\mathrm{cut}
$$

In this case, a step S1 will be applied if R1 is not a logical rule that introduces A, otherwise S2 will be applied. In the S1 case, the right derivation will be carried to the top, without modifying m and a. Clearly, this can happen multiple times, but in each of these residues, the situation remains unchanged. In the S2 case, on the other hand, the derivation π_1 (weakened in some axioms with Γ and Δ) will be taken and substituted instead of the (ax) rule. However, in the residue of a, A is main in a logical rule R1, because the cut is of the S2 type. This is the only case where the rule m changes, replaced by another rule where A is still the main conclusion. As can be seen, the fact that A is the main conclusion in the new rule is given by the definition of the S1 and S2 cuts. □

Theorem 3 (Focusing Stability)**.** *Let π be a focused proof to which a $\mathcal{T}_{glob}$ transformation is applied, resulting in the derivation π'. Then, π' is still focused.*

Proof. The proof is trivial by looking at the previous lemma. □

Once the stability is proven, we then move on to demonstrate completeness, that is, from a certain non-focused proof π, by following a certain procedure, one can obtain a focused proof π' with the same concluding sequent.

Theorem 4 (Completeness of focusing)**.** *Each proof π in $LK_{\forall\to}$ which is not focused can be transformed in a focused proof with the same conclusion.*

Proof. Since we are dealing with completeness, we will consider a derivation system where the only admitted cut is the particular form of cut with a formula of the theory. Indeed, thanks to the cut-elimination theorem, we know that the system where the only form of cut elimination is the one discussed, is complete. We call *forbidden* any irreversible rule or cut having as active formula an irreversible formula which is not main. The proof is by induction on the number of forbidden rules of π. We select a forbidden rule such that there is no other forbidden rule above it. The case of the cut rule is straightforward, as when the active formula F is not the main conclusion of the previous rule, we can permute the cut rule with the previous one. In other words, in the case of the cut, having the active formula in the cut as the main in the previous rule means having a logical cut: by following the cut elimination procedure, every cut can be transformed into a logical cut (or be removed)[14].

The other forbidden rules can be either $(\to L)$ or $(\forall L)$ and the left formulas are of the form $A \to B$, $\forall xC$, or A atomic formula. Therefore, in total, we need to address six possible cases.

Let us now underline, once and for all, that—to demonstrate what we propose—we will cut the derivation π with other derivations. These derivations are not necessarily carried out in $LK_{\forall\to}$. Indeed, the axioms are applied to possibly non-atomic formulas. However, the derivations are admissible in standard *LK* and, as we will see later, they will not appear in the final focused derivation, which will indeed be in $LK_{\forall\to}$. The core idea is that one takes a forbidden sub-derivation, transforms it, and then reinserts it. In doing so, new forbidden rules cannot be created because the last rule of the transformed derivation coincides.

$A \to B$ **active in $(\to L)$**

We suppose that $(R1)$ is not a $(\to L)$ rule with main conclusion $A \to B$ and we can suppose that $(\to L)$ it is the last rule of the derivation:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, A \to B \vdash \Delta}\,(R1) \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Sigma \vdash C, \Lambda}\,(R2)}{\Gamma, \Sigma, C \to (A \to B) \vdash \Delta, \Lambda}\,(\to L)$$

We cut our derivation with the following derivation of $C \to (A \to B) \vdash C \to (A \to B)$, obtaining a new derivation with the same conclusion:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{C \vdash C}\,(ax) \qquad \dfrac{\dfrac{}{A \vdash A}\,(ax) \qquad \dfrac{}{B \vdash B}\,(ax)}{A \to B, A \vdash B}\,\to L}{C \to (A \to B), A, C \vdash B}\,\to L}{C \to (A \to B), C \vdash A \to B}\,\to R}{C \to (A \to B) \vdash C \to (A \to B)}\,\to R \qquad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, A \to B \vdash \Delta}\,(R1) \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Sigma \vdash C, \Lambda}\,(R2)}{\Gamma, \Sigma, C \to (A \to B) \vdash \Delta, \Lambda}\,(\to L)}{\Gamma, \Sigma, C \to (A \to B) \vdash \Delta, \Lambda}\,(cut)$$

We proceed to eliminate the cut according to the standard procedure, in particular this is a case of logical cut followed by a cut on C:

[14] In this case, the observation made in the previous section is of fundamental importance. If an axiom introduced by the unary cut rule does not become main in its history in a logical rule or an ax rule, then it is eliminable.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Sigma \vdash C, \Lambda}\,(\mathrm{R2})
      \qquad
      \dfrac{\dfrac{}{A \vdash A}\,(\mathrm{ax}) \quad \dfrac{}{B \vdash B}\,(\mathrm{ax})}{A \to B, A \vdash B}\,{\to}\mathrm{L}
    }{\Sigma, C \to (A \to B), A \vdash B, \Lambda}\,{\to}\mathrm{L}
  }{\Sigma, C \to (A \to B) \vdash A \to B, \Lambda}\,{\to}\mathrm{R}
  \qquad
  \dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, A \to B \vdash \Delta}\,(\mathrm{R1})
}{\Gamma, \Sigma, C \to (A \to B) \vdash \Delta, \Lambda}\,(\mathrm{cut})
$$

Since $A \to B$ is not the main conclusion of (R1), using commutative reduction steps and duplicating if required the sub-derivation with conclusion $\Sigma, C \to (A \to B) \vdash A \to B, \Lambda$, we trace back in π_1 every introduction of $A \to B$ through ($\to$L)[15].

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Sigma \vdash C, \Lambda}\,(\mathrm{R2})
      \qquad
      \dfrac{A \vdash A \quad B \vdash B}{A \to B, A \vdash B}\,{\to}\mathrm{L}
    }{\Sigma, C \to (A \to B), A \vdash B, \Lambda}\,{\to}\mathrm{L}
  }{\Sigma, C \to (A \to B) \vdash A \to B, \Lambda}\,{\to}\mathrm{R}
  \qquad
  \dfrac{
    \begin{matrix}\pi_1'\\ \vdots\\ \Gamma_1 \vdash A, \Delta_1\end{matrix}
    \quad
    \begin{matrix}\pi_1''\\ \vdots\\ \Gamma_2, B \vdash \Delta_2\end{matrix}
  }{\Gamma_1, \Gamma_2, A \to B \vdash \Delta_1, \Delta_2}\,({\to}\mathrm{L})
}{\Gamma, \Sigma, C \to (A \to B) \vdash \Delta, \Lambda}\,(\mathrm{cut})
$$

We proceed to eliminate the cut, obtaining the following rule:

$$
\dfrac{
  \dfrac{
    \begin{matrix}\pi_1'\\ \vdots\\ \Gamma_1 \vdash A, \Delta_1\end{matrix}
    \quad
    \begin{matrix}\pi_1''\\ \vdots\\ \Gamma_2, B \vdash \Delta_2\end{matrix}
  }{\Gamma_1, \Gamma_2, A \to B \vdash \Delta_1, \Delta_2}\,({\to}\mathrm{L})
  \qquad
  \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Sigma \vdash C, \Lambda}\,(\mathrm{R2})
}{\Gamma_1, \Gamma_2, \Sigma, C \to (A \to B) \vdash \Delta_1, \Delta_2, \Lambda}\,({\to}\mathrm{L})
$$

Now notice that in the proof π' thus obtained the unique forbidden rule of π has become a certain number $n \geq 0$ of rules that are not forbidden anymore (remember that due to the choice of the forbidden rule in π there are no other forbidden rules above it).

$\forall xC$ **active in ($\to$L)**

We suppose that $(R1)$ is not a ($\forall$L) rule with main conclusion $\forall xC$ and we can suppose that ($\to$L) it is the last rule of the derivation:

$$
\dfrac{
  \dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, \forall xC \vdash \Delta}\,(\mathrm{R1})
  \qquad
  \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Sigma \vdash A, \Lambda}\,(\mathrm{R2})
}{\Gamma, \Sigma, A \to \forall xC \vdash \Delta, \Lambda}\,({\to}\mathrm{L})
$$

We cut our derivation with the following derivation of $A \to \forall xC \vdash A \to \forall xC$, obtaining a new derivation with the same conclusion:

[15] All cases where $A \to B$ has not been introduced by a logical rule are handled normally, following the cut elimination procedure.

\begin{prooftree}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$A \vdash A$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$C(y/x) \vdash C(y/x)$}
\RightLabel{$\forall$L}
\UnaryInfC{$\forall x C \vdash C(y/x)$}
\RightLabel{$\to$L}
\BinaryInfC{$A \to \forall x C, A \vdash C(y/x)$}
\RightLabel{$\forall$R}
\UnaryInfC{$A \to \forall x C, A \vdash \forall x C$}
\RightLabel{$\to$R}
\UnaryInfC{$A \to \forall x C \vdash A \to \forall x C$}
\AxiomC{π_1}
\noLine
\UnaryInfC{$\vdots$}
\RightLabel{(R1)}
\UnaryInfC{$\Gamma, \forall x C \vdash \Delta$}
\AxiomC{π_2}
\noLine
\UnaryInfC{$\vdots$}
\RightLabel{(R2)}
\UnaryInfC{$\Sigma \vdash A, \Lambda$}
\RightLabel{($\to$L)}
\BinaryInfC{$\Gamma, \Sigma, A \to \forall x C \vdash \Delta, \Lambda$}
\RightLabel{(cut)}
\BinaryInfC{$\Gamma, \Sigma, A \to \forall x C \vdash \Delta, \Lambda$}
\end{prooftree}

We proceed to eliminate the cut according to the standard procedure, in particular this is a case of the elimination of logical cut followed by an elimination on the cut on A:

\begin{prooftree}
\AxiomC{π_2}
\noLine
\UnaryInfC{$\vdots$}
\RightLabel{(R2)}
\UnaryInfC{$\Sigma \vdash A, \Lambda$}
\AxiomC{$C(y/x) \vdash C(y/x)$}
\RightLabel{$\forall$L}
\UnaryInfC{$\forall x C \vdash C(y/x)$}
\RightLabel{$\to$L}
\BinaryInfC{$\Sigma, A \to \forall x C \vdash \Lambda, C(y/x)$}
\RightLabel{$\forall$R}
\UnaryInfC{$\Sigma, A \to \forall x C \vdash \Lambda, \forall x C$}
\AxiomC{π_1}
\noLine
\UnaryInfC{$\vdots$}
\RightLabel{(R1)}
\UnaryInfC{$\Gamma, \forall x C \vdash \Delta$}
\RightLabel{(cut)}
\BinaryInfC{$\Gamma, \Sigma, A \to \forall x C \vdash \Delta, \Lambda$}
\end{prooftree}

Since $\forall x C$ is not the main conclusion of (R1), using commutative reduction steps and duplicating if required the subderivation with conclusion $\Sigma, A \to \forall x C \vdash \Lambda, \forall x C$, we trace back in π_1 every introduction of $\forall x C$ through ($\forall$R):

\begin{prooftree}
\AxiomC{π_2}
\noLine
\UnaryInfC{$\vdots$}
\RightLabel{(R2)}
\UnaryInfC{$\Sigma \vdash A, \Lambda$}
\AxiomC{$C(y/x) \vdash C(y/x)$}
\RightLabel{$\forall$L}
\UnaryInfC{$\forall x C \vdash C(y/x)$}
\RightLabel{$\to$L}
\BinaryInfC{$\Sigma, A \to \forall x C \vdash \Lambda, C(y/x)$}
\RightLabel{$\forall$R}
\UnaryInfC{$\Sigma, A \to \forall x C \vdash \Lambda, \forall x C$}
\AxiomC{π_1'}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma_1, C \vdash \Delta_1$}
\RightLabel{($\forall$L)}
\UnaryInfC{$\Gamma_1, \forall x C \vdash \Delta_1$}
\RightLabel{(cut)}
\BinaryInfC{$\Gamma, \Sigma, A \to \forall x C \vdash \Delta, \Lambda$}
\end{prooftree}

We proceed to eliminate the cut, obtaining the following rule

\begin{prooftree}
\AxiomC{π_1'}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma_1, C \vdash \Delta_1$}
\RightLabel{($\forall$L)}
\UnaryInfC{$\Gamma_1, \forall x C \vdash \Delta_1$}
\AxiomC{π_2}
\noLine
\UnaryInfC{$\vdots$}
\RightLabel{(R2)}
\UnaryInfC{$\Sigma \vdash A, \Lambda$}
\RightLabel{($\to$L)}
\BinaryInfC{$\Gamma_1, \Sigma, C \to (A \to B) \vdash \Delta_1, \Lambda$}
\end{prooftree}

Now notice that in the proof π' thus obtained the unique forbidden rule of π has become a certain number $n \geq 0$ of rules that are not forbidden anymore (remember that due to the choice of the forbidden rule in π there are no other forbidden rules above it).

$A \to B$ **active in (∀L)**

We suppose that $(R1)$ is not a (→L) rule with main conclusion $A \to B$ and we can suppose that (∀L) it is the last rule of the derivation:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, A \to B \vdash \Delta}\,(R1)}{\Gamma, \forall x(A \to B) \vdash \Delta}\,(\forall L)$$

We cut our derivation with the following derivation of $\forall x(A \to B) \vdash \forall x(A \to B)$, obtaining a new derivation with the same conclusion:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax) \quad \dfrac{}{B \vdash B}\,(ax)}{A \to B, A \vdash B}\,\to L}{\forall x(A \to B), A \vdash B}\,\forall L}{\forall x(A \to B) \vdash A \to B}\,\to R}{\forall x(A \to B) \vdash \forall x(A \to B)}\,\forall R \qquad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, A \to B \vdash \Delta}\,(R1)}{\Gamma, \forall x(A \to B) \vdash \Delta}\,(\forall L)}{\Gamma, \forall x(A \to B) \vdash \Delta}\,(cut)$$

We proceed to eliminate the cut according to the standard procedure, in particular this is a case of logical cut.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax) \quad \dfrac{}{B \vdash B}\,(ax)}{A \to B, A \vdash B}\,\to L}{\forall x(A \to B), A \vdash B}\,\forall L}{\forall x(A \to B) \vdash A \to B}\,\to R \qquad \dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, A \to B \vdash \Delta}\,(R1)}{\Gamma, \forall x(A \to B) \vdash \Delta}\,(cut)$$

We now trace back each introduction of $A \to B$ and proceed as in the previous cases.

$\forall xC$ **active in** $\forall L$

We suppose that $(R1)$ is not a (∀L) rule with main conclusion $\forall xC$ and we can suppose that (∀L) it is the last rule of the derivation:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, \forall xC \vdash \Delta}\,(R1)}{\Gamma, \forall y \forall xC \vdash \Delta}\,(\forall L)$$

We cut our derivation with the following derivation of $\forall y \forall xC \vdash \forall y \forall xC$, obtaining a new derivation with the same conclusion:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{C \vdash C}\,(\mathrm{ax})}{\forall x C \vdash C}\,\forall \mathrm{L}}{\forall y \forall x C \vdash C}\,\forall \mathrm{L}}{\forall y \forall x C \vdash \forall x C}\,\forall \mathrm{R}}{\forall y \forall x C \vdash \forall y \forall x C}\,\forall \mathrm{R}
\qquad
\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, \forall x C \vdash \Delta}\,(\mathrm{R1})}{\Gamma, \forall y \forall x C \vdash \Delta}\,(\forall \mathrm{L})
}{\Gamma, \forall y \forall x C \vdash \Delta}\,(\mathrm{cut})
$$

In this case as well, we first eliminate the logical cut and then eliminate the S1 cut.

B **(atomic) active in** $(\rightarrow L)$

We suppose that $(R1)$ is not a (ax) rule with main conclusion B and we can suppose that (→L) it is the last rule of the derivation:

$$
\dfrac{
\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, B \vdash \Delta}\,(\mathrm{R1})
\qquad
\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Sigma \vdash A, \Lambda}\,(\mathrm{R2})
}{\Gamma, \Sigma, A \rightarrow B \vdash \Delta, \Lambda}\,(\rightarrow \mathrm{L})
$$

We cut our derivation with the following derivation of $A \rightarrow B \vdash A \rightarrow B$, obtaining a new derivation with the same conclusion:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{ax}) \quad \dfrac{}{B \vdash B}\,(\mathrm{ax})}{A \rightarrow B, A \vdash B}\,\rightarrow \mathrm{L}}{A \rightarrow B \vdash A \rightarrow B}\,\rightarrow \mathrm{R}
\qquad
\dfrac{
\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, B \vdash \Delta}\,(\mathrm{R1})
\quad
\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Sigma \vdash A, \Lambda}\,(\mathrm{R2})
}{\Gamma, \Sigma, A \rightarrow B \vdash \Delta, \Lambda}\,(\rightarrow \mathrm{L})
}{\Gamma, \Sigma, A \rightarrow B \vdash \Delta, \Lambda}\,(\mathrm{cut})
$$

We proceed to eliminate the cut according to the standard procedure. In particular, this is the only case where the introduced derivation will be partially present in the final derivation. However, this is not a problem because B is atomic. The case $(\forall L)$ is analogous. □

2.2.3. Reversion

Definition 15 (Reverted proof)**.** *We say that a cut-free focused proof in $LK_{\forall,\rightarrow}$ is reverted if every sequent in the proof that contains a reversible formula is the conclusion of a reversible rule.*

The idea is that one can see a proof (reading the tree bottom-up) as an alternation of sequences of reversible rules and sequences of irreversible rules (this has been strongly exploited for example in (Girard 2001)). Indeed, if one starts with a sequent with reversible formulas one can decide to apply reversible rules until this is possible, then choose an irreversible formula and apply a sequence of irreversible rules following the focusing constraint and start again the procedure.

Lemma 6 (Completeness of Reversion)**.** *Every focused proof can be transformed into a reverted proof.*

Proof. In addressing the completeness of reversion, as was the case with Focusing, we will consider a system without cuts, except for those cuts involving formulas of the theory **T**.

We denote by $\mu(F)$ the complexity of the formula, *i.e.* the total number of connectives and quantifiers. We introduce a size $\mu_r(\pi)$ on a derivation π, measuring how far from a reverted proof π is. The definition is by induction on $l(\pi)$, the number of rules of π. We denote by $\mu(\Gamma)$ the sum of $\mu(A)$ for A a formula of Γ. Let R be the last rule of π: we call π_1 (resp. π_1 and π_2) the subproof(s) whose sequent(s) conclusion is (resp. are) premise(s) of R.

- if R is an axiom rule with conclusion $\Gamma, A \vdash A, \Delta$, we set $\mu_R(\pi) = \mu(\Delta)$ (remember A is atomic);
- if R is a cut rule, we set: $\mu_r(\pi) = \mu_r(\pi_1) + \mu(\Delta)$.
- if R is a left contraction rule, we set: $\mu_r(\pi) = \mu_r(\pi_1) + \mu(\Delta)$.
- if R is a right contraction rule, we set: $\mu_r(\pi) = \mu_r(\pi_1) + \mu(\Delta) + \mu(A)$.
- if $R = (\forall L)$, we set: $\mu_r(\pi) = \mu_r(\pi_1) + \mu(\Delta)$.
- if $R = (\rightarrow L)$, we set: $\mu_r(\pi) = \mu_r(\pi_1) + \mu_r(\pi_2) + \mu(\Delta, \Lambda)$.
- if $R = (\rightarrow R)$, we set: $\mu_r(\pi) = \mu_r(\pi_1)$.
- if $R = (\forall R)$, we set: $\mu_r(\pi) = \mu_r(\pi_1)$.

Notice that, for every focused derivation π, one has $\mu_r(\pi) = 0 \iff \pi$ is a reverted derivation. We prove that any focused derivation π can be transformed into a reverted derivation π^r, by induction on the pair $(\mu_r(\pi), l(\pi))$ lexicographically ordered. We analyze all the possible cases for the last rule R of π.

ax

$$\frac{}{\Gamma, A \vdash A, \Delta}\ (\text{ax})$$

In the set Δ, there are possibly non-atomic formulas, while we recall that A is necessarily atomic. However, we note that every reversible rule preserves contexts: in particular—looking bottom-up—the fragment $A \vdash A$ cannot disappear if only reversible rules are applied. This means that reversible rules can be executed until obtaining a Δ' such that $\mu(\Delta') = 0$, and at that point, the axiom rule can be executed.

Reversible Rule

$$\frac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma' \vdash \Delta'\end{matrix}}{\Gamma \vdash \Delta}\ \text{rev}$$

Since $l(\pi_1) < l(\pi)$ and $\mu_r(\pi) = \mu_r(\pi_1)$, by induction, there exists a reverted π_1^r that derives $\Gamma' \vdash \Delta'$. Therefore, the following derivation is reverted.

$$\dfrac{\begin{array}{c}\pi_1^r\\ \vdots\\ \Gamma' \vdash \Delta'\end{array}}{\Gamma \vdash \Delta}\ \text{rev}$$

Irreversible Rule

We only deal with the case of the rule ($\forall$L). The discussion of ($\to$L) follows the same reasoning.

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ A(t/x), \Gamma \vdash \Delta\end{array}}{\forall x A(x), \Gamma \vdash \Delta}\ (\forall\text{L})$$

For IH, there exists a reverted derivation π_1^r that derives the same sequent conclusion $A(t/x), \Gamma \vdash \Delta$.

$$\dfrac{\begin{array}{c}\pi_1^r\\ \vdots\\ A(t/x), \Gamma \vdash \Delta\end{array}}{\forall x A(x), \Gamma \vdash \Delta}\ (\forall\text{L})$$

Let us now suppose that $\mu(\Delta) > 0$ with $\forall z B(z) \in \Delta$ (the case where $B \to C \in \Delta$ is analogous).

We trace back in π_1^r the introduction of $\forall z B(z)$. Since π_1^r is reverted, the introduction of $\forall z B(z)$ is not only unique, but necessarily given by a logical rule.

$$\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\pi_1^{r'}\\ \vdots\\ \Gamma' \vdash B(y), \Delta'\end{array}}{\Gamma' \vdash \forall z B(z), \Delta'}\ (\forall\ \text{R})\\ \vdots\\ A(t/x), \Gamma \vdash \Delta\end{array}}{\forall x A(x), \Gamma \vdash \Delta}\ (\forall\text{L})$$

Therefore, we consider the derivation π_1^r without the rule ($\forall$ R), applying the rule at the end of the derivation instead.

$$\dfrac{\dfrac{\begin{matrix}\pi_1^{r''}\\ \vdots \\ A(t/x),\Gamma\vdash B(y),\Delta''\end{matrix}}{\forall xA(x),\Gamma\vdash B(y),\Delta''}\,(\forall L)}{\forall xA(x),\Gamma\vdash\Delta}\,(\forall R)$$

The same procedure can be applied with all reversible formulas in Δ, until obtaining $\mu(\Delta)=0$, that is, a reverted derivation.

CR

Let us deal with one of the more important cases, namely when R is a right contraction on a reversible formula $C\to D$: the proof π can be represented as follows

$$\dfrac{\begin{matrix}\pi_1\\ \vdots \\ \Gamma\vdash C\to D, C\to D,\Delta\end{matrix}}{\Gamma\vdash C\to D,\Delta}$$

We have that $\mu_r(\pi)=\mu_r(\pi_1)+\mu(C\to D)+\mu(\Delta)>\mu_r(\pi_1)$. By induction hypothesis π_1 can be transformed into a reverted proof π_1^r that necessarily has the following shape:

$$\begin{matrix}
\dfrac{\vdots}{\Gamma_1,C,C\vdash\Delta_1}\ \mathrm{R}_1 \text{ not reversible}\\
\vdots\\
\dfrac{\Gamma_{11},C,C\vdash D,\Delta_{11}}{\Gamma_{11},C\vdash C\to D,\Delta_{11}}\to_1\\
\vdots\\
\dfrac{\Gamma_{12},C\vdash D,C\to D,\Delta_{12}}{\Gamma_{12}\vdash C\to D,C\to D,\Delta_{12}}\to_2\\
\vdots\\
\Gamma\vdash C\to D,C\to D,\Delta
\end{matrix}$$

where all the rules of π_1^r following R_1 (which is itself not reversible) are reversible rules. We can then call π_2^r the derivation obtained from π_1^r by simply erasing the two rules $\to_1$ and $\to_2$: this derivation is clearly still reverted and its sequent conclusion is $\Gamma,C,C\vdash D,D,\Delta$. Let π_3 be the derivation with conclusion $\Gamma,C,C\vdash D,\Delta$ obtained from π_2^r by performing a contraction on the right with main conclusion D: we have by definition that $\mu_r(\pi_3)=\mu_r(\pi_2^r)+\mu(D)+\mu(\Delta)=\mu(D)+\mu(\Delta)<\mu_r(\pi_1)+\mu_r(C\to D)+\mu_r(\Delta)=\mu_r(\pi)$. We can thus apply the induction hypothesis to π_3: there exists a reverted proof π_3^r with conclusion $\Gamma,C,C\vdash D,\Delta$. Let π_4 be the derivation with conclusion $\Gamma,C\vdash D,\Delta$ obtained from π_3^r by performing a contraction on the left with main conclusion C: we have by definition that $\mu_r(\pi_4)=\mu_r(\pi_3^r)+\mu(D)+\mu(\Delta)=\mu(D)+\mu(\Delta)<\mu_r(\pi_1)+\mu_r(C\to D)+\mu_r(\Delta)=\mu_r(\pi)$. And we can again apply the induction hypothesis, this time to π_4: there exists a reverted proof π_4^r with conclusion $\Gamma,C\vdash D,\Delta$. The reverted derivation π^r is then obtained by applying the reversible rule $\to R$ to the conclusion of π_4^r: this indeed yields a reverted proof with conclusion $\Gamma \vdash C \to D, \Delta$.

CL

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma, A, A \vdash \Delta\end{array}}{\Gamma A \vdash \Delta}\ (\mathrm{CL})$$

Since $l(\pi_1) < l(\pi)$, then there exists a reverted derivation π_1^r such that

$$\dfrac{\begin{array}{c}\pi_1^r\\ \vdots\\ \Gamma, A, A \vdash \Delta\end{array}}{\Gamma A, \vdash \Delta}\ (\mathrm{CL})$$

If the CL is not admissible, this means that the last rule of π_1^r is a reversible rule $\tilde{R}$.

$$\dfrac{\begin{array}{c}\pi_1^{r'}\\ \vdots\\ \Gamma', A, A \vdash \Delta'\end{array}}{\Gamma, A, A \vdash \Delta}\ \mathrm{rev}\ \tilde{R}$$

where $\mu(\Delta') < \mu(\Delta)$

Let us now consider the derivation π^*.

$$\dfrac{\begin{array}{c}\pi_1^{r'}\\ \vdots\\ \Gamma', A, A \vdash \Delta'\end{array}}{\Gamma', A \vdash \Delta'}\ (\mathrm{CL})$$

We note that $\mu_r(\pi^*) = \mu_r(\pi_1^r) + \mu(\Delta') = \mu(\Delta') < \mu(\Delta) \leq \mu(\pi)$.

By inductive hypothesis, there thus exists a reverted derivation that proves $\Gamma', A \vdash \Delta'$. If the rule $\tilde{R}$ is added as the last rule of this derivation, the sought-after derivation is obtained.

cut

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma, F \vdash \Delta\end{array}}{\Gamma \vdash \Delta}\ (\mathrm{cut})$$

Since $l(\pi_1) < l(\pi)$, then there exists a reverted derivation π_1^r that derives $\Gamma, F \vdash \Delta$.

$$\dfrac{\begin{array}{c}\pi_1^r\\ \vdots\\ \Gamma, F \vdash \Delta\end{array}}{\Gamma \vdash \Delta}\ (\text{cut})$$

Let's now assume that the cut is inadmissible, meaning that $\mu(\Delta) > 0$. Consequently, the last rule of the derivation π_1^r must be a reversible one.

$$\dfrac{\begin{array}{c}\pi_1^{r'}\\ \vdots\\ \Gamma', F \vdash \Delta'\end{array}}{\Gamma \vdash \Delta}\ \text{rev } \tilde{R}$$

where $\mu(\delta') < \mu(\delta)$.

Let us now consider the following derivation π^*.

$$\dfrac{\begin{array}{c}\pi_1^{r'}\\ \vdots\\ \Gamma', F \vdash \Delta'\end{array}}{\Gamma' \vdash \Delta'}\ (\text{cut})$$

$\mu_r(\pi^*) = \mu_r(\pi_1^r) + \mu(\Delta) = \mu_r(\Delta') + \mu(\Delta) < \mu(\pi)$.

By inductive hypothesis, there thus exists a reverted derivation that proves $\Gamma' \vdash \Delta'$. If the rule $\tilde{R}$ is added to this derivation, the desired derivation is obtained. □

Before proceeding to discuss the stability of reversion with respect to cut elimination, it is necessary to make some observations. As we have seen in Chapter 1, the reversion constraint proposed by the $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ game (*i.e.*, the Opponent's move rule) appears to be more stringent than the one proposed in Definition 15. Indeed, once a reversible formula is chosen, the Opponent is obliged to expand that formula as much as possible, and even if other formulas appear in $\mathscr{V}$, Opponent is compelled to stay focused on that one. In other words, the Opponent also seems to be subject to a kind of "focusing" constraint. However, introducing this additional requirement into the sequent calculus is not complicated and, as the following lemma shows, in a reverted proof this corresponds to having at most one reversible formula in the sequent conclusion.

Lemma 7. *In a reverted proof where the sequent conclusion has at most one reversible formula, every occurrence of a reversible formula is the main conclusion of a logical rule.*

Proof. Let's start by noting that, in a reverted proof where there is at most one reversible formula in the concluding sequent, in every sequent of the proof the number of reversible formulas is always at most one. Indeed, referring to the rules introduced in 2.2, the only rules that increase (reading the derivation bottom-up) the number of formulas present in the right part of the sequent are (CR) and (→L). However, since the proof is reverted, the CR cannot have a reversible formula as its active formula. Furthermore, since the proof is reverted, in the rule (→L), Γ and Λ contain only atomic formulas: consequently, even in $\Gamma \vdash \Delta, A$ there will be at most one reversible formula.

Therefore, if there is at most one reversible formula in the sequent conclusion of the derivation, this will be true for every sequent in the derivation. This means that every sequent in the proof either has exactly one reversible formula or has no reversible formulas. From the reversion constraint, every sequent that contains a reversible formula is the conclusion of a reversible rule, and since in every sequent the reversible formula is unique, it is definitely the main one. □

In light of these observations, we are now ready to give the definition of a *fully reverted* proof. As can be seen from the definition, in a fully reverted derivation—unlike a reverted proof—the use of cut is allowed.

Definition 16 (Fully Reverted)**.** *We say that a focused proof in $LK_{\forall,\to}$ is fully reverted if the following conditions are satisfied:*

1. *every sequent has at most one reversible formula;*
2. *each reversible formula is either main in a logical rule or in the conclusion of a cut.*

Remark 2. *If we were considering only cut-free derivations, it would suffice to require that every reversible formula is main, and to achieve this in a reverted proof, it would be enough to impose that in the sequent conclusion there is at most one reversible formula (see Lemma 7).*

However, since cuts are allowed in a fully reverted proof, it is necessary to require that every sequent has at most one reversible formula. To understand why, consider the following example, where both A and B are reversible formulas, while the formula C active in the cut is an atomic formula.

$$\dfrac{\Gamma, C \vdash A, \Delta \qquad \Sigma \vdash C, B, \Lambda}{\Gamma, \Sigma \vdash A, B, \Delta, \Lambda}\ (cut)$$

A derivation containing a cut like this does not break the second requirement of the definition of fully reverted, because, with A and B appearing in different sequents, both can then be main. Therefore, requirement number one has been added to the definition of a fully reverted proof, without which the constraint of being fully reverted would not be stable for cut elimination.

Clearly, the definition of Fully Reverted does not prevent a reversible formula from being active in a cut rule.

If one limits oneself to derivations that have at most one reversible formula in the concluding sequent, then the constraint of being fully reverted is complete, because every reverted proof that has at most one reversible formula in the sequent conclusion is fully reverted. However, we emphasize that this restriction is relatively natural: since we are interested in the derivability of a formula F, it is not restrictive to consider those derivations where the sequent conclusion is of the form $\vdash F$.

Moreover, as we will see in the following lemma, in any reverted proof, after a possible initial block (looking bottom-up) of reversible rules—possibly applied to different formulas without any "focusing" constraint—it remains true that each sequent contains at most one reversible formula.

Lemma 8. *Every reverted derivation is a cut-free fully reverted derivation with the addition of a certain number of reversible rules at the end.*

Proof. Let π be a reverted proof with sequent conclusion $\Gamma \vdash \Delta$, where Δ contains various reversible formulas. The reversion constraint mandates, from bottom to top, to apply reversible formulas until the sequent $\Gamma' \vdash \Delta'$ is reached, in which Δ' consists exclusively of atomic formulas. At this point, $\Gamma' \vdash \Delta'$ becomes the sequent conclusion of a proof to which the hypotheses of Lemma 7 can be applied. □

We also note that the fully reversion constraint is a *local constraint*: in other words, it is enough to observe a single rule—including its premises and conclusion—to determine whether that rule violates the reversion constraint. This is generally not the case for focusing, because by observing the formulas in the conclusion, we cannot determine which one was active in the previous rule.

We will therefore show the stability of reversion only for fully reverted proofs, which corresponds to the $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ game scenario.

Before moving forward, let us make an important observation on the structural step S_2 in light of the reversion constraint.

Remark 3. *In a reverted proof with cuts, the right occurrence of the active formula in any cut, provided it is not atomic, could not have been—throughout its history—the main conclusion of a contraction. That is to say, the introduction of every reversible formula is unique and occurs only through a logical rule.*

Thus, if we have a cut where the right occurrence of the active formula A is not main, the sole possibility is that A is in the context of the sequent conclusion of another cut, as illustrated in the following derivation, with A in Λ_1 or Λ_2.

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, A \vdash \Delta}\,R1 \qquad \dfrac{\dfrac{\begin{matrix}\pi_{2'}\\ \vdots\end{matrix}}{\Sigma_1, B \vdash \Lambda_1} \qquad \dfrac{\begin{matrix}\pi_{2''}\\ \vdots\end{matrix}}{\Sigma_1 \vdash B, \Lambda_2}}{\Sigma \vdash A, \Lambda}\,(cut)}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,(cut)$$

Let us notice that, if B is a reversible formula, A must necessarily belong to Λ_1.

In other words, between the introduction of a reversible formula A and the moment when A becomes active (in a logical rule or in a cut), there can only be a certain number of cuts where A is part of the context.

Lemma 9 (Full Reversion Stability)**.** *Let π be a fully reverted proof to which a $\mathscr{T}_{glob}$ step is applied, obtaining the derivation π'. Then, π' is still reverted.*

Proof. In the proof, we focus on the rules that precede the cut we aim to eliminate.

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, A \vdash \Delta}\,\text{R1} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Sigma \vdash A, \Lambda}\,\text{R2}}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,(\text{cut})$$

We note that, in general, in Λ there cannot be any reversible formulas.

Let us assume that the (cut) is a logical cut, with both rules introducing the implication.

$$\dfrac{\dfrac{\begin{matrix}\pi_1'\\ \vdots\\ \Gamma \vdash \Delta, B\end{matrix} \qquad \begin{matrix}\pi_1''\\ \vdots\\ C, \Gamma' \vdash \Delta'\end{matrix}}{\Gamma, \Gamma', B \to C \vdash \Delta, \Delta'}\,(\to\text{L}) \qquad \dfrac{\begin{matrix}\pi_2'\\ \vdots\\ B, \Sigma \vdash \Lambda, C\end{matrix}}{\Sigma \vdash B \to C, \Lambda}\,(\to\text{R})}{\Gamma, \Gamma', \Sigma \vdash \Delta, \Delta', \Lambda}\,(\text{cut})$$

Being the derivation reverted, neither Δ, nor Δ', nor Λ can contain reversible formulas. By performing a cut elimination step, the derivation transforms into the following:

$$\dfrac{\dfrac{\begin{matrix}\pi_2'\\ \vdots\\ B, \Sigma \vdash \Lambda, C\end{matrix} \qquad \begin{matrix}\pi_1'\\ \vdots\\ \Gamma \vdash \Delta, B\end{matrix}}{\Gamma, \Sigma \vdash \Delta, \Lambda, C}\,(\text{cut}) \qquad \begin{matrix}\pi_1''\\ \vdots\\ C, \Gamma' \vdash \Delta'\end{matrix}}{\Sigma, \Gamma, \Gamma' \vdash \Delta, \Delta', \Lambda}\,(\text{cut})$$

As can be easily seen, the proof is still reverted.

Let us now analyze the case where both rules introduce the 'for all' quantifier.

$$\dfrac{\dfrac{\begin{matrix}\pi_1'\\ \vdots\\ \Gamma, C[t/x] \vdash \Delta\end{matrix}}{\Gamma, \forall x C(x) \vdash \Delta}\,(\forall\text{L}) \qquad \dfrac{\begin{matrix}\pi_2'\\ \vdots\\ \Sigma \vdash C[y/x], \Lambda\end{matrix}}{\Sigma \vdash \forall x C(x), \Lambda}\,(\forall\text{R})}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,(\text{cut})$$

Then a cut elimination step transforms it into the following derivation:

$$\dfrac{\begin{matrix}\pi_1'\\ \vdots\\ \Gamma, C[t/x] \vdash \Delta\end{matrix} \qquad \begin{matrix}\pi_2'\\ \vdots\\ \Sigma \vdash C[y/x], \Lambda\end{matrix}}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,(\text{cut})$$

In this case as well, the derivation remains reverted.

To address the structural step S2, referring to Lemma 3, it suffices to verify that the constraint of full reversion is maintained during a (CC) step. More specifically, S2 applies a certain number of (CC) steps to make two cuts commute until a logical cut is obtained[16].

Let A be the active formula in the cut on which S2 is working, and let us assume that A is in the sequent conclusion of another cut with B as active formula. We also assume that B is a reversible formula.

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma, A \vdash \Delta}\,\mathrm{R1} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_{2'}\\ \vdots\end{array}}{\Sigma_1, B \vdash A, \Lambda_1} \quad \dfrac{\begin{array}{c}\pi_{2''}\\ \vdots\end{array}}{\Sigma_1 \vdash B, \Lambda_2}}{\Sigma \vdash A, \Lambda}\,(\mathrm{cut})}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,(\mathrm{cut})$$

Let us apply the (CC) step, obtaining the following derivation.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma, A \vdash \Delta}\,\mathrm{R1} \quad \dfrac{\begin{array}{c}\pi_{2'}\\ \vdots\end{array}}{\Sigma_1, B \vdash A, \Lambda_1}}{\Gamma, \Sigma, B \vdash \Lambda_1}\,(\mathrm{cut}) \qquad \dfrac{\begin{array}{c}\pi_{2''}\\ \vdots\end{array}}{\Sigma_1 \vdash B, \Lambda_2}}{\Gamma, \Sigma \vdash \Delta, \Lambda}\,(\mathrm{cut})$$

As can be seen, the derivation remains fully reverted.

Let us now address the case of the structural step S1.

The introduction of A could have occurred either through a logical rule or as context of an axiom rule (ax). In the case of a logical rule, the (CC) step is applied until reaching the S2 case, just addressed.

In the case of the introduction of A as context of the axiom rule (ax), the (CC) step is applied until obtaining the following derivation:

$$\dfrac{\dfrac{}{\Gamma, A, C \vdash C, \Delta}\,(\mathrm{ax}) \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Sigma \vdash A, \Lambda}\,\mathrm{R2}}{\Gamma, \Sigma, C \vdash C, \Delta, \Lambda}\,(\mathrm{cut})$$

This cut is eliminated by removing the derivation π_2 and weakening, in the axiom rule (ax), with Σ and Λ. Since, as mentioned at the beginning, there are no reversible formulas in Λ, the proof remains reverted. $\square$

2.2.4. LK$_{\mathbf{game}}$

Let us see how the rules of LK$_{\forall\to}$ can be rewritten in light of the properties of focusing and reversion. In the following system, we will also assume that *each sequent contains*

[16] We recall, in fact, that in the S2 case, the introduction of the left occurrence of the cut formula is logical.

only formulas written in Krivine normal form. From the viewpoint of provability, this choice is not restrictive, as shown in Chapter 1.

The basic idea is that we can view a derivation as an alternation of focusing blocks (rule L) and reversion blocks (rule R), possibly interspersed with contractions.

Structural Rules

$$\frac{F, F, \Gamma \vdash \Delta}{F, \Gamma \vdash \Delta}\ (\mathrm{CL}) \qquad\qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ (\mathrm{CR}) \text{ where } A \text{ is atomic}$$

Identity and Logical Rules

$$\frac{\Gamma_1 \vdash F_1(\vec{t}/\vec{x}), \Delta_1 \quad \ldots \quad \Gamma_n \vdash F_n(\vec{t}/\vec{x}), \Delta_n}{\Gamma \vdash \Delta, A(\vec{t}/\vec{x})}\ (\mathrm{L}) \text{ with } F \in \Gamma \cup \mathbf{T}$$

$$\frac{\Gamma, F_1(\vec{y}/\vec{x}), \ldots, F_n(\vec{y}/\vec{x}) \vdash \Delta, A(\vec{y}/\vec{x})}{\Gamma \vdash \forall \vec{x}(F_1(\vec{x}), \ldots, F_n(\vec{x}) \to A(\vec{x})), \Delta}\ (\mathrm{R})$$

In the (L) rule, $F = \forall \vec{x} F_1, \ldots, F_n \to A$[17], $\Gamma \setminus F = \bigcup_{i=1}^{n} \Gamma_i$, and $\Delta = \bigcup_{i=1}^{n} \Delta_i$ consists of atomic formulas only.

Let us notice that (L), in the case of $n = 0$, is the (ax) rule. Moreover, always with respect to the rule (L), the formula F on which the focusing block finished could be in Γ (logical rule) or in $\mathbf{T}$ (cut).

Before we can define the LK_{game} system, we need to make some further modifications to our system. Indeed, we aim to better capture the essence of the proof-search fashion of the $\mathbf{T}\mathcal{UVA}$ game: while in the $\mathbf{T}\mathcal{UVA}$ game, every position reached from a winning position for **P** is still winning for **P**, the same is not true for sequents (in a bottom-up reading of the tree). For example, the wrong constants may have been instantiated with the rule $(\forall L)$, or a formula that would later be useful in a certain branch of the derivation was not contracted.

To address this issue, we need two key components: ensuring that no formula in the context is lost and that no main formula is lost. Furthermore, since we know from reversion that right contraction can only be performed on atomic formulas, and that right atomic formulas are "main", besides in CR, only in (L), during the rule (R) there's no need to copy the main formula, even from a proof-search perspective.

We will therefore provide a procedure to transform a reverted proof into what we will define as a *canonical proof.* This proof structure will induce even stricter rules, suitable for proof-search, which we will define as LK_{game}.

Lemma 10. *Let π be a fully reverted proof. Each occurrence of (L) rule in π can be transformed into*

$$\frac{\Gamma \vdash F_1(\vec{t}/\vec{x}), \Delta, A(\vec{t}/\vec{x}) \quad \ldots \quad \Gamma \vdash F_n(\vec{t}/\vec{x}), \Delta, A(\vec{t}/\vec{x})}{\Gamma \vdash \Delta, A(\vec{t}/\vec{x})}\ (\mathrm{L})^* \text{ with } F \in \Gamma \cup \boldsymbol{T}$$

[17] We are using here the notation introduced in Chapter 1.

Proof. We first note that by adding appropriate contexts to (ax) rules, each rule (L) can be transformed into

$$\frac{\Gamma \setminus \{F\} \vdash F_1(\vec{t}/\vec{x}), \Delta, A(\vec{t}/\vec{x}) \quad \dots \quad \Gamma \setminus \{F\} \vdash F_n(\vec{t}/\vec{x}), \Delta, A(\vec{t}/\vec{x})}{\Gamma, \Gamma \setminus \{F\}, \dots, \Gamma \setminus \{F\} \vdash \Delta, \dots, \Delta, A(\vec{t}/\vec{x})} \text{ (L) with } F \in \Gamma \cup \mathbf{T}$$

Subsequently, by making the appropriate contractions, the sequent conclusion can be transformed into the following sequent (we recall that in Δ there are only atomic formulas):

$$\dfrac{\dfrac{\Gamma \setminus \{F\} \vdash F_1(\vec{t}/\vec{x}), \Delta, A(\vec{t}/\vec{x}) \quad \dots \quad \Gamma \setminus \{F\} \vdash F_n(\vec{t}/\vec{x}), \Delta, A(\vec{t}/\vec{x})}{\Gamma, \Gamma \setminus \{F\}, \dots, \Gamma \setminus \{F\} \vdash \Delta, \dots, \Delta, A(\vec{t}/\vec{x})} \text{ (L)* with } F \in \Gamma \cup \mathbf{T}}{\Gamma \vdash \Delta, A(\vec{t}/\vec{x})} \text{ (CL)s and (CR)s}$$

We note that the sequent conclusion $\Gamma \vdash \Delta, A(\vec{t}/\vec{x})$ is the same as that of rule (L).

Furthermore, before starting (reading the derivation bottom-up) each focusing block, the main conclusion of the focusing block can be contracted, obtaining the rule

$$\frac{\Gamma \vdash F_1(\vec{t}/\vec{x}), \Delta, A(\vec{t}/\vec{x}) \quad \dots \quad \Gamma \vdash F_n(\vec{t}/\vec{x}), \Delta, A(\vec{t}/\vec{x})}{\Gamma \vdash \Delta, A(\vec{t}/\vec{x})} \text{ (L)* with } F \in \Gamma \cup \mathbf{T}$$

Finally, we note that the contractions imposed by the *canonical proof* are sufficient to obtain a complete system: indeed, any formula present in the left context or in the atomic right context of any sequent is present throughout the subderivation that has that sequent as its root. □

We are thus ready to define the system LK$_{game}$.

Definition 17 (LK$_{game}$)**.** *In the (L) rule, $F = \forall \vec{x}(F_1(\vec{x}), \dots, F_n(\vec{x}) \to A(\vec{x})) \in \Gamma \cup \boldsymbol{T}$ with $n \geq 0$. In particular, when $n = 0$, (L) is an axiom.*

$$\frac{\Gamma \vdash F_1(\vec{t}/\vec{x}), \Delta, A(\vec{t}/\vec{x}) \quad \dots \quad \Gamma \vdash F_n(\vec{t}/\vec{x}), \Delta, A(\vec{t}/\vec{x})}{\Gamma \vdash \Delta, A(\vec{t}/\vec{x})} \textit{ (L)}$$

$$\frac{\Gamma, F_1(\vec{y}/\vec{x}), \dots, F_n(\vec{y}/\vec{x}) \vdash \Delta, A(\vec{y}/\vec{x})}{\Gamma \vdash \forall \vec{x}(F_1(\vec{x}), \dots, F_n(\vec{x}) \to A(\vec{x})), \Delta} \textit{ (R)}$$

Theorem 5 (Completeness of LK$_{game}$)**.** *The sequent $\Gamma \vdash \Delta$ is provable in LK$_{\forall,\to}$ if and only if it is provable in LK$_{game}$*

Proof. The rules of LK$_{game}$can be simulated by a certain number of rules of LK$_{\forall,\to}$, so that one can prove, by induction on the number of rules of the derivation π of LK$_{game}$ that every sequent conclusion of a rule of π can be derived in LK$_{\forall,\to}$.

Conversely, every derivation in $LK_{\forall,\to}$ can be transformed into a canonical derivation through the procedure proposed in Lemma 10. □

Lemma 11 (Substitution)**.** *If $\Gamma \vdash \Delta$ is provable using a proof tree of height n, then for any substitution τ (a function that maps free variables to terms), the sequent $\tau(\Gamma) \vdash \tau(\Delta)$ is also provable using a proof tree of the same height.*

Lemma 12 (Generic constants)**.** *If $\vec{g}$ are constant symbols, pairwise distinct, and they do not appear in Γ nor Δ, then the following rule is admissible, where $\vec{y}$ are fresh variables.*

$$\frac{\Gamma, G_1(\vec{g}), \ldots, G_n(\vec{g}) \vdash \Delta, B(\vec{g})}{\Gamma \vdash \forall \vec{y}(G_1(\vec{y}/\vec{g}), \ldots, G_n(\vec{y}/\vec{g}) \to B(\vec{y}/\vec{g})), \Delta}\ (R^*)$$

2.2.5. The $\perp$ Connective

We note that up to now, the role of falsehood, *i.e.*, $\perp$, has not been discussed. This is because, although it is essential to establish particular rules for the connective $\to$ and for the quantifier $\forall$, the same does not apply for the connective $\perp$. As we will discover, rather than defining specific left and right rule, for $\perp$, it will be sufficient to introduce an assumption regarding the sequent conclusion.

From now on, we assume the presence in our language of constant relation $\perp$. In $LK_{\forall\to}$ this constant has no special status. We also consider the derivation system $LK_{\forall\to\perp}$ where sequents are the same as in $LK_{\forall\to}$ but $\perp$ has a special status given by its usual rules[18]:

$$\frac{}{\Gamma, \perp \vdash \Delta}\ (\perp \mathrm{L}) \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \perp}\ (\perp \mathrm{R})$$

It is well known that a sequent $\Gamma \vdash \Delta$ in Gentzen's LK if and only if $\Gamma^* \vdash \Delta^*$ is provable in $LK_{\forall\to\perp}$ where Λ^* is the sequent obtained by a suitable translation (see for example (Krivine and Legrandgérard 2007)) of sequent Λ written in full classical language.

Lemma 13. *$\Gamma \vdash \Delta$ is provable in $LK_{\forall\to\perp}$ if and only if $\Gamma \vdash \Delta, \perp$ is provable in $LK_{\forall\to}$.*

Proof. If $\Gamma \vdash \Delta, \perp$ is provable in $LK_{\forall\to}$ with the proof tree π then we obtain the following derivation of $\Gamma \vdash \Delta$ in $LK_{\forall\to\perp}$ by means of the cut rule.

$$\frac{\dfrac{}{\perp \vdash}\ (\perp\mathrm{L}) \qquad \begin{array}{c}\pi \\ \vdots \\ \Gamma \vdash \Delta, \perp\end{array}}{\Gamma \vdash \Delta}\ (\mathrm{cut})$$

Conversely, suppose that $\Gamma \vdash \Delta$ is the sequent conclusion of a derivation π of LKftb. We prove that π can be transformed into a derivation of $LK_{\forall\to}$ with conclusion $\Gamma \vdash$

[18]We recall that weakening is included in axioms.

$\Delta, (\bot)^n$ for some $n \geq 1$ (then by a contraction rule we obtain a proof of $\Gamma \vdash \Delta, \bot$ as required). By Gentzen's cut-elimination theorem, we can suppose π to be cut-free. We explain how every occurrence of a (left or right) rule for $\bot$ can be removed and a $\bot$ formula can be added to the conclusion. More formally, the proof is by induction on the number of rules for $\bot$ in π:

- if in π there is no rule for $\bot$ (neither right nor left), then π is itself a derivation of $LK_{\forall\to}$ and we can pick an (arbitrary) axiom of π and add a formula $\bot$ on the right in its sequent conclusion. We thus obtain a derivation of $LK_{\forall\to}$with conclusion $\Gamma \vdash \Delta, \bot$;

- above a rule $(\bot R)$, there is at least one axiom or one rule $(\bot L)$: in both cases we can pick one of these rules and change it into an axiom rule of $LK_{\forall\to}$ with two more $\bot$ on the right of the sequent conclusion and erase the rule $(\bot R)$: the sequent conclusion of this new derivation π' is $\Gamma \vdash \Delta, \bot$ and π' has one less rule for $\bot$;

- a rule $(\bot L)$ is a leaf of the derivation tree π with conclusion, say, $\Sigma, \bot, \vdash \Lambda$: we can substitute such a rule by the axiom rule of $LK_{\forall\to}$ with conclusion $\Sigma, \bot, \vdash \Lambda, \bot$. The sequent conclusion of this new derivation π' is $\Gamma \vdash \Delta, \bot$ and π' has one less rule for $\bot$. $\square$

3. Winning strategies as Proofs

In this chapter, we show the equivalence, under certain conditions, between a winning strategy for the Proponent in the game $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ and a derivation in $\mathrm{LK}_{\mathrm{game}}$.

In the following statements, we assume a theory $\mathbf{T}$ is given and provability is understood within this theory.

Lemma 14. *If $\mathscr{U} \vdash \mathscr{A}$ is provable in LK_{game}then the position $(\mathscr{U},\mathscr{A})$ is winning for* $\mathbf{P}$.

Proof. We prove it by induction on the tree height of a proof of $\mathscr{U} \vdash \mathscr{A}$. Since $\mathscr{A}$ only contains atomic formulas by definition, the last rule must be (L), so the proof has the following shape, where $F \in \mathscr{U} \cup \mathbf{T}$ and $F(\vec{t})_0 \in \mathscr{A}$:

$$\dfrac{\cdots \quad \begin{array}{c}\pi_i\\ \vdots\\ \mathscr{U} \vdash F(\vec{t})_i, \mathscr{A}\end{array} \quad \cdots}{\mathscr{U} \vdash \mathscr{A}}\ (\mathrm{L})$$

We now consider the move $(\mathscr{U},\mathscr{A}) \xrightarrow{F,\vec{t}} (\mathscr{U},\mathscr{V},\mathscr{A})$ with $\mathscr{V} = \{F(\vec{t})_1, \ldots, F(\vec{t})_n\}$. By Lemma 2, it suffices to prove that $(\mathscr{U},\mathscr{V},\mathscr{A})$ is winning for $\mathbf{P}$, for which we apply Lemma 3. Consider a $\mathbf{O}$-move $(\mathscr{U},\mathscr{V},\mathscr{A}) \xrightarrow{F(\vec{t})_i,\vec{u}} (\mathscr{U}',\mathscr{A}')$ with $\mathscr{U}' = \mathscr{U} \cup \{F(\vec{t})_i(\vec{u})_1, \ldots, F(\vec{t})_i(\vec{u})_m\}$ and $\mathscr{A}' = \mathscr{A} \cup \{F(\vec{t})_i(\vec{u})_0\}$).

If $F(\vec{t})_i$ is atomic, then $\mathscr{U}' \vdash \mathscr{A}'$ is the conclusion of π_i, hence $(\mathscr{U}',\mathscr{A}')$ is winning for $\mathbf{P}$ by induction. Otherwise, π_i must have the shape

$$\dfrac{\begin{array}{c}\pi_i'\\ \vdots\\ \mathscr{U}, F(\vec{t})_i(\vec{x})_1, \ldots, F(\vec{t})_i(\vec{x})_m \vdash \mathscr{A}, F(\vec{t})_i(\vec{x})_0\end{array}}{\mathscr{U} \vdash F(\vec{t})_i, \mathscr{A}}\ (\mathrm{R})$$

Observe that $\mathscr{U}' \vdash \mathscr{A}'$ is the conclusion of π_i' where $\vec{u}$ is substituted for $\vec{x}$. By Lemma 11, $\mathscr{U}' \vdash \mathscr{A}'$ is provable with a tree of the same height as π_i' so induction applies, from which we deduce that $(\mathscr{U}',\mathscr{A}')$ is winning for $\mathbf{P}$. Therefore $(\mathscr{U}',\mathscr{A}')$ is winning for $\mathbf{P}$ in any case, from which we can conclude. $\square$

Corollary 1. *If $\mathscr{U} \vdash F, \mathscr{A}$ is provable in $LK_{game}$for each $F \in \mathscr{V}$ then the position $(\mathscr{U},\mathscr{V},\mathscr{A})$ is winning for* $\mathbf{P}$.

The converse, however, does not hold in general. Suppose, for example, that the set of closed terms in our language is finite, $t_1, \ldots, t_n$. $\mathbf{P}$ has a winning strategy for the formula $F = \forall x(P(t_1),...,P(t_n) \to P(x))$, where P is a unary predicate: the winning strategy is simply to pick up $P(t_i)$ from $\mathscr{U}$ whenever **O** chooses t_i as x. On the other hand, F is not provable: as can be seen in the following proof, it is not possible to proceed with the derivation with any rule.

$$\frac{P(t_1),...,P(t_n) \vdash P(y)}{\vdash \forall x(P(t_1),...,P(t_n) \to P(x))}\ (\mathrm{R})$$

The fact that the strategy above is winning relies on the fact that, from the point of view of the game, no elements exist besides $t_1, \ldots, t_n$ since the players have no way to name them, while this hypothesis does not apply in the proof system (unless some axiom of the theory states it). The idea is that if we can have control on which closed terms **O** can play, there could be winning strategies which do not correspond to proofs. In order to have a proper match between proofs and strategies, **O** should be always able to play *generic* elements.

Definition 18 (fresh constant)**.** *A constant c is* fresh *with respect to a theory **T** (resp. with respect to a position in the game for **T**) if it does not occur in any axiom of **T** (nor in any formula in that position).*

For the definition below, we suppose that the language $\mathscr{L}$ has countably many constant symbols and is equipped with an enumeration $(c_n)_{n\in\mathbf{N}}$ for them.

Definition 19 (generic **O** move)**.** *An* **O** *move $(F,\vec{b})$ from a position $(\mathscr{U},\mathscr{V},\mathscr{A})$ is called generic if $\vec{b}$ consists of the first items in sequence (c_n) that are fresh with respect to $(\mathscr{U},\mathscr{V},\mathscr{A})$.*

Note that a generic move is completely defined by the choice of F in $\mathscr{V}$. Therefore, since $\mathscr{V}$ is always finite, there are finitely many generic moves from each **O** position.

Definition 20 (σ-size)**.** *Let σ be a* **P** *strategy. The σ-size $|a|_\sigma$ of a position a is the upper bound of the lengths of all plays starting from P that respect strategy σ and in which all* **O** *moves are generic. The size is either a natural number or ∞.*

Lemma 15. *If the strategy σ is winning for a position a, then $|a|_\sigma < \infty$.*

Proof. Since σ is winning for P, each possible σ-play starting from P is finite. Consider the tree of all σ-plays from P that use generic **O** moves only. This tree is finitely branching, because **P** positions all have exactly one move (given by σ) and **O** positions have finitely many possible moves as remarked above. Since all branches are finite, because P is winning, we get by König's lemma that the height of the tree is finite. Hence the set of length of considered plays is bounded, which means that $|P|_\sigma$ is a natural number. □

For the next two statements, we assume that the language contains infinitely many constants that are fresh for the theory **T**.

Lemma 16. *If a position $(\mathscr{U},\mathscr{A})$ is winning for* **P** *then $\mathscr{U} \vdash \mathscr{A}$ is provable in LK_{game}*

Proof. Let σ be a strategy for **P**. We prove that for each position $(\mathscr{U},\mathscr{A})$ for which σ is winning, there exists a proof of $\mathscr{U}\vdash\mathscr{A}$, by induction on $|(\mathscr{U},\mathscr{A})|_\sigma$.

Consider a move $(\mathscr{U},\mathscr{A})\xrightarrow{F,\vec{b}}(\mathscr{U},\mathscr{V},\mathscr{A})$ in σ, with $\mathscr{V}=\{F(\vec{b})_1,\dots,F(\vec{b})_n\}$. We prove that for each i, the sequent $\mathscr{U}\vdash F(\vec{b})_i,\mathscr{A}$ is provable, from which we will conclude by the rule

$$\frac{\mathscr{U}\vdash F(\vec{b})_1,\mathscr{A}\qquad\dots\qquad\mathscr{U}\vdash F(\vec{b})_n,\mathscr{A}}{\mathscr{U}\vdash\mathscr{A}}\ (\mathrm{L})$$

For each i, since there are infinitely many constants that are fresh for **T**, we know that there exists a generic **O**-move $(\mathscr{U},\mathscr{V},\mathscr{A})\xrightarrow{F(\vec{b})_i,\vec{c}}(\mathscr{U}',\mathscr{A}')$, where $\mathscr{U}'=\mathscr{U}\cup\{F(\vec{b})_i(\vec{c})_1,\dots,F(\vec{b})_i(\vec{c})_k\}$ and $\mathscr{A}'=\mathscr{A}\cup\{F(\vec{b})_i(\vec{c})_0\}$. Besides, since σ is winning from $(\mathscr{U},\mathscr{A})$, it is also winning from $(\mathscr{U}',\mathscr{A}')$ and $|(\mathscr{U}',\mathscr{A}')|_\sigma<|(\mathscr{U},\mathscr{A})|_\sigma$. By induction hypothesis, we deduce that there exists a proof π_i of $\mathscr{U},F(\vec{b})_i(\vec{c})_1,\dots,F(\vec{b})_i(\vec{c})_k\vdash\mathscr{A},F(\vec{b})_i(\vec{c})_0$ and by construction the constants in $\vec{c}$ do not occur in $\mathscr{U}$ or $\mathscr{A}$. By Lemma 12, we can thus deduce a proof π'_i where these constants are replaced by a sequence $\vec{x}$ of fresh variables, then we can apply the (R) rule

$$\frac{\begin{array}{c}\pi'_i\\ \vdots\\ \mathscr{U},F(\vec{b})_i(\vec{x})_1,\dots,F(\vec{b})_i(\vec{x})_k\vdash\mathscr{A},F(\vec{b})_i(\vec{x})_0\end{array}}{\mathscr{U}\vdash F(\vec{b})_i,\mathscr{A}}\ (\mathrm{R})$$

from which we can conclude. □

Corollary 2. *If a position $(\mathscr{U},\mathscr{V},\mathscr{A})$ is winning for* **P** *then $\mathscr{U}\vdash F,\mathscr{A}$ is provable in $LK_{game}$for every $F\in\mathscr{V}$.*

Theorem 6. *Assuming that infinitely many constants are fresh for* **T***, a sequent $\vdash F$ is provable in $LK_{\forall,\to}$ if and only if the position $(\emptyset,\{F\},\emptyset)$ is winning for* **P**.

Proof. If $\vdash F$ is provable in $\mathrm{LK}_{\forall,\to}$, then it is provable in $\mathrm{LK}_{\mathrm{game}}$ hence by 1 the position $(\emptyset,\{F\},\emptyset)$ is winning for **P**. Reciprocally, if $(\emptyset,\{F\},\emptyset)$ is winning for **P**, then by 2 the sequent $\vdash F$ is provable in $\mathrm{LK}_{\mathrm{game}}$ hence also in $\mathrm{LK}_{\forall,\to}$. □

Therefore, recalling the discussion on the role of falsehood, made in section 2.2.5, we have the following theorem.

Theorem 7. *Assuming that infinitely many constants are fresh for* **T***, a sequent $\vdash F$ is provable in $LK_{\forall\to\bot}$ if and only if the position $(\emptyset,\{F\},\{\bot\})$ is winning for* **P**.

Proof. By Lemma 13, $\vdash F$ is provable in $\mathrm{LK}_{\forall\to\bot}$ if and only if $\vdash F,\bot$ is provable in $\mathrm{LK}_{\forall\to}$. By Corollary 2 and Corollary 1, $\vdash F,\bot$ is provable in $\mathrm{LK}_{\forall\to}$ if and only if the position $(\emptyset,\{F\},\{\bot\})$ is winning for **P**. □

It is interesting to wonder what difference there is in having **O** play with specific constants or generic constants (*i.e.*, variables) at an epistemological level. We will return to this issue in section 5.2.3.4.

3.1. Strategies *vs* Models

This section heavily relies on the result obtained by Krivine (2006). However, in one of the two directions of the theorem, a hypothesis is weakened, resulting in a more general result.

Definition 21 (false position). *A* **P***-position* $(\mathscr{U},\mathscr{A})$ *is false in a structure* $\mathscr{M}$ *if* $\mathscr{M}\vDash F$ *for all* $F\in\mathscr{U}$ *and* $\mathscr{M}\nvDash A$ *for all* $A\in\mathscr{A}$*. An* **O***-position* $(\mathscr{U},\mathscr{V},\mathscr{A})$ *is false in* $\mathscr{M}$ *if* $\mathscr{M}\vDash F$ *for all* $F\in\mathscr{U}$*,* $\mathscr{M}\nvDash A$ *for all* $A\in\mathscr{A}$ *and* $\mathscr{M}\nvDash F$ *for at least one* $F\in\mathscr{V}$*.*

Before getting into the details of the proof, we remark that we can assume that each language $\mathscr{L}$ has at least one constant. That is because models are non-empty, so it will always be possible to extend our language with a constant without changing anything.

Lemma 17. *If* F *is valid then* **P** *has a winning strategy over* F*.*

Proof. Let $(\Psi_n)_n$ be a sequence of all possible normal formulas in $\mathscr{L}$, and let $(c_n)_n$ be a sequence of all closed terms in our language, where repetitions are allowed. We consider a standard Cantor enumerate argument to build a suitable sequence $(\Psi,c)_n$ that lists all combinations. We define a **P** strategy σ this way. The Proponent plays the first allowed Ψ,c in the sequence which has not been played before. We show that if this strategy is not winning, we build a counter-model $\mathscr{M}$, where $\mathscr{M}$ is the model the satisfies all and exactly all the closed atomic formulas that are never put in $\mathscr{A}$ during the game. We define $(\Psi,\vec{b})$ *acceptable* if, at same point during the game, $\Psi\in\mathscr{U}$ and $\Psi(\vec{b})_0\in\mathscr{A}$. It is readily seen that if σ is loosing, every acceptable move is indeed played at same point during the game. Indeed, let $(\Psi,\vec{b})$ acceptable but not played. At a certain point during the game, $\Psi\in\mathscr{U}$ and $\Psi(\vec{b})_0\in\mathscr{A}$ and all acceptable moves before it have been played (since it is the first counter-example). At this point, the strategy tells to play $(\Psi,\vec{b})$ and **P** can do so. As proved in (Krivine 2006) by induction, the model $\mathscr{M}$ defined above satisfies all formulas in $\mathscr{U}$ and the negation of every formula chosen by **O** during the game. □

3.1.1. Counter-models as Opponent strategies

In the following when we refer to a model $\mathscr{M}$ we a refer to a countable model for a theory **T** over a language $\mathscr{L}$. Moreover, we assume that the interpretation function restricted to closed terms is surjective. In other words, every element in $\mathscr{M}$ is the interpretation of at least one constant in $\mathscr{L}$.

In general, assuming that if a formula is not valid then there is a countable model which does not satisfy it is wrong. But, as soon as we restrict ourselves to countable languages, it works. On the other hand, countable languages are enough for most mathematical areas.

Lemma 18. *Assume* ***T*** *has infinitely many fresh constants. If a position* $(\mathscr{U},\mathscr{V},\mathscr{A})$ *is false in some model of* ***T*** *then it is winning for* **O***.*

Proof. Let S be the set of positions that are false in some model of $\mathscr{M}$. It suffices to prove that there is a **O**-strategy σ that is total over S and such that each play that starts from a position in S stays in S.

By definition, if $(\mathscr{U}, \mathscr{V}, \mathscr{A})$ is false in some model $\mathscr{M}$, then all formulas in $\mathscr{U}$ are true in $\mathscr{M}$, all formulas in $\mathscr{A}$ are false in $\mathscr{M}$ and some formula $\forall x_1 \dots x_k\, F_1, \dots, F_n \to A$ in $\mathscr{V}$ is false in $\mathscr{M}$. Thus there exists a valuation of each x_i by some point α_i in $\mathscr{M}$ that satisfies $F_1, \dots, F_n$ and falsifies A. Consider constants $c_1, \dots, c_k$ fresh for $\mathbf{T}$ and $(\mathscr{U}, \mathscr{V}, \mathscr{A})$. Define a structure $\mathscr{M}'$ as $\mathscr{M}$ except that each c_i is interpreted by α_i. Then by construction $\mathscr{M}' \vDash F_i[\vec{c}/\vec{x}]$ for each i and $\mathscr{M}' \nvDash A[\vec{c}/\vec{x}]$. Hence $(\mathscr{U} \cup \{F_i[\vec{c}/\vec{x}] \mid i = 1, \dots, n\}, \mathscr{A} \cup \{A[\vec{c}/\vec{x}]\})$ is false in $\mathscr{M}'$. $\square$

Notably, this chapter provides a new proof of the *Completeness Theorem* through winning strategies.

4. Composition of Strategies

In this chapter, we discuss the composition of strategies, specifically how strategies can *interact* to form a new strategy. This is analogous to what happens in the interaction of proofs through cut-elimination.

4.1. A Clear Idea That Doesn't Work

In this first section, we show a possible path to take for the composition of strategies. Although it turns out to be a dead end, it is useful as preliminary insight before reading the following section.

The general purpose of this chapter is the following: we have two winning strategies σ, on the formula $A \to B$, and τ, on the formula $B \to C$, and we want to construct a winning strategy υ on the formula $A \to C$. In other words, σ is a winning strategy for **P** in the position $(A, B, \emptyset)$ while τ is a winning strategy for **P** in the position $(B, C, \emptyset)$.

More precisely, during the play starting from the position $(A, C, \emptyset)$, **P** consults and makes σ and τ interact to know what move to make at each turn. The first to play is **O**, who chooses constants $\vec{a}$ to be substituted to the variables of the first level quantifier of C (see Figure 4.1).

turn	$\mathscr{U}$	$\mathscr{V}$	$\mathscr{A}$	move
O	A	C		$C, \vec{a}$
P	$A, C\vec{a}_*$		$C\vec{a}_0$	

Figure 4.1.

At this point, in the position $(\{A, C\vec{a}_*\}, C\vec{a}_0)$, **P** must decide what move to make, and to do so, begins an *interaction game*. That is, **P** does on (B, C) the same move that **O** made on (A, C), obtaining the position $(\{B, C\vec{a}_*\}, C\vec{a}_0)$ (see Figure 4.3). If, now, τ chooses one of the $C\vec{a}_*$, **P** simply redo the move chosen by τ in its game against **O**. However, if τ chooses the move $(B, \vec{b})$, then we have a type of move that we will later call an *internal move*: **P** reports the move made by τ on the other component $(A, B, \emptyset)$, obtaining the position $(\{A, B\vec{b}_*\}, B\vec{b}_0)$ (see Figure 4.2). Similarly, if σ chooses A, the move is copied into the main game against **O**; otherwise, the process continues with another *internal move*.

Summarizing and referring to Figure 4.4, if one of the two strategies plays on the cut formula B or on a subformula of it, then an internal move is performed (L or R in

turn	$\mathscr{U}$	$\mathscr{V}$	$\mathscr{A}$	move
O	A	B		$B, \vec{b}$
P	$A, B\vec{b}_*$		$B\vec{b}_0$	...

Figure 4.2.: The first component of the interaction, which is in a quiet state until a move is made on B in the second component.

turn	$\mathscr{U}$	$\mathscr{V}$	$\mathscr{A}$	move
O	B	C		$C, \vec{a}$
P	$B, C\vec{a}_*$		$C\vec{a}_0$	$B, \vec{b}$

Figure 4.3.: The second component of the interaction, where **P** copies the move $(C, \vec{a})$ made by **O** in the main play.

the figure), that is, a move not copied in the main game. If another formula is played instead, the move is copied in the main game ($\mathbf{P}_1$ or $\mathbf{P}_2$) and then the subsequent move by **O** ($\mathbf{O}_1$ or $\mathbf{O}_2$) is copied into the interaction game. The Q indicates a **O** position in a quiet state, waiting for an internal move.

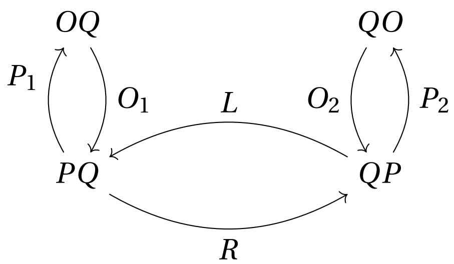


Figure 4.4.: Types of positions and moves in the interaction game.

For clarity, referring to Figure 4.5, if you cross the line π, then the move is made in the main game; if you cross the line π_1, the move is made in the first component; and if you cross the line π_2, the move is made in the second component. With each move, two lines are necessarily crossed.

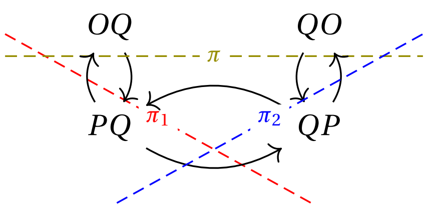


Figure 4.5.: Interaction positions and their projections.

Unfortunately, although the core idea is correct, the procedure does not work. Indeed, as can be seen, for example, from Figure 4.2, after B has been played once, it disappears from $\mathscr{V}$: if τ were to play again on $B \in \mathscr{U}$ at some point, this move could not be replicated in the first component. Indeed, it should not be too surprising that the procedure, as it stands, does not work, since our goal is to compose strategies, not plays.

4.2. An Involved Idea That (might) Work

As can be easily inferred from the title of this section, the work we will present here is still in progress. The main idea is that various strategies will interact, each starting from a specific position, to obtain a new strategy. Unlike the previous section, we will define interaction in general, not just limited to the case of two strategies.

Let us then proceed to reformulate and generalize the idea just presented. In the following definitions, I represents the set of positions where a strategy exists, while p is the main position (as shown in Figure 4.1 in the previous section) where a new strategy is to be constructed from the strategies in I. R_I is an interaction relation that will prove useful later on.

Definition 22 (Composed Position)**.** *A composed position is a triple* $p = (p, I, R_I)$ *where*

- p *is a position, referred to as* main position[1]*;*
- I *is a finite sequence of couples of positions and strategies, i.e.* $I = (i_i, s_i)_{i\in\{0,\ldots,n\}}$ *where* i_i *is a position and* s_i *a strategy.*
- R_I *is a binary relation over the set of positions* i_i

We will write $i_k \in I_k$ *meaning* $i \in (i_k, s_k)$*. We define the function* len *such that* $len(I) = n$*.*

In the following definition, we will call $\mathscr{U}_x$ (resp. $\mathscr{V}_x$, $\mathscr{A}_x$) the $\mathscr{U}$ set (resp. $\mathscr{V}$, $\mathscr{A}$) of the position x.

Definition 23 (Interaction Position)**.** *A composed position* p *it's called an interaction position if the following applies:*

- *If* p *is a* **O***-position then* i_i *is an* **O***-position for all* $i \in \{0,\ldots,n\}$
- *If* p *is a* **P***-position then there exists a unique* k *such that* i_k *is a* **P***-position*
- *For every formula* F *in* $\mathscr{U}_p$*, (resp.* $\mathscr{V}_p$*) there exist a not necessarily unique* k *such that* $F \in \mathscr{U}_{i_k}$ *(resp.* $F \in \mathscr{V}_{i_k}$*)*
- $\mathscr{A}_p = \bigcup_{i=1}^n A_{i_i}$

Definition 24 (Interaction Move in a sequence)**.** *Let* π *be a sequence of interaction positions. A move* m *in* π *is a couple* $m_\pi = (F, \vec{b})$ *such that:*

[1] p is the main position where **P** and **O** play.

- *An* **O***-move goes from a* **O***-position to a* **P***-position. It is a couple* $(F, \vec{b})$ *where* $F \in \mathcal{V}_{p_i}$ *and goes from the position* p_i *to the position* $p' = ((\mathcal{U}_{p'} + F\vec{b}_*, \mathcal{A}_{p'} + F\vec{b}_0), \overline{I})$ *where* $\overline{I}$ *has the same* (i_i, s_i) *of* I *but for one* k *for which* $i_k = (\mathcal{U}_{i_k} + F\vec{b}_*, \mathcal{A}_{i_k} + F\vec{b}_0)$.

- *A* **P***-move goes from a* **P***-position to a* **O***-position. It is a couple* $(F, \vec{b})$ *where* $F \in \mathcal{U}_{p_i}$ *and* $F\vec{b}_0 \in \mathcal{A}_{p_i}$. *It goes from the position* p_i *to the position* $p_{i+1} = ((\mathcal{U}_{p_i}, F\vec{b}_*, \mathcal{A}_{p_i}), \overline{I})$ *where* $\overline{I}$ *has the same* (i_i, s_i) *of* I *but for the unique* k *such that* i_k *is a* **P***-position where* $i_k = (\mathcal{U}_{i_k}, F\vec{b}_*, \mathcal{A}_{i_k})$.

- *A* i*-move (internal) goes from a* **P***-position to a* **P***-position. It is a couple* $(F, \vec{b})$ *where* $F \in \mathcal{U}_{i_k}$ *and* $F\vec{b}_0 \in \mathcal{A}_{i_k}$, *where* i_k *is the only* **P***-position. It goes from the position* p_i *to the position* p_{i+1} *such that:*

$$\begin{cases} \textbf{\textit{type 1}}\ p_{i+1} = (p_i, I_1) & \text{if } F \in V_{i_j} \text{ for some } j \\ \textbf{\textit{type 2}}\ p_{i+1} = (p_i, I_2) & \text{otherwise} \end{cases}$$

  where

  - I_1 *is the same as* I *but for the two components* $i_k = (\mathcal{U}_{i_k}, F\vec{b}_*, \mathcal{A}_{i_k})$ *and* $i_j = (\mathcal{U}_{i_j} + F\vec{b}_*, \mathcal{A}_{i_k} + F\vec{b}_0)$. *The two new positions* i_k *and* i_j *thus obtained are added to the interaction relation, in other words* $(i_k, i_j) \in R_I$.

  - I_2 *is the same as* I *but for the component* $i_k = (\mathcal{U}_{i_k}, F\vec{b}_*, \mathcal{A}_{i_k})$. *A new component* $n+1$ *is added where, given* $i_t \in \pi$ *such that* F *in* $\mathcal{V}_{i_t}$, $i_{n+1} = (\mathcal{U}_{i_t} + Fb_*, \mathcal{A}_{i_t} + Fb_0)$ *and* $s_{n+1} = s_t$. *The two new positions* i_k *and* i_{n+1} *thus obtained are added to the interaction relation, in other words* $(i_k, i_{n+1}) \in R_I$.

As can be noted, in the previous definition the relation R_I does not impose any constraints on the moves that can or cannot be made. The relation serves solely to relate positions obtained from the same i-move. The R_I relation will then prove useful in limiting the possibilities for interaction that can occur among the various components of I.

Definition 25 (Initial Interaction Position)**.** *An interaction position is a initial interaction position if for all formulas* F *which are not in* $\mathcal{U}_p$ *but are in* $\mathcal{U}_{i_k}$ *for some* k, $F \in V_{i_j}$ *for some* $j \neq k$. R_I *is initially empty.*

Definition 26 (Interaction play)**.** *An interaction play* π *is a sequence* $(p_0, p_1, \ldots, p_n)$ *of interaction positions such that*

- p_0 *is an* Initial Interaction Position

- *for each* $i > 0$, p_i *is the result of a move in* $(p_0, \ldots, p_{i-1})$, *as of Definition 24.*

We will write $p_i <_\pi p_j$ *if* $i < j$.

To understand the situation better we can have a look at the following diagram, which illustrates the three possible kind of moves between the two possible kind of positions.

$$\begin{tikzcd} O \arrow[d, bend right, "O"'] \\ P \arrow[u, bend right, "P"'] \arrow[loop below, "i"] \end{tikzcd}$$

Remark 4. *Even though the moves are correctly defined, the definition do not highlight a fundamental aspect. Indeed, at first glance,* **O***-moves and* **P***-moves might seem like actions that occur in the main component p and are then replicated, albeit simultaneously, in the set I. It is this very simultaneity that overlooks a key aspect which the reader should keep in mind moving forward: an* **O***-move is chosen in p and emulated in I, while a* **P***-move is chosen in I and emulated in p.*

The next Lemma 19 ensures that every move made within an internal component of I effectively leads to an interaction move.

Lemma 19. *Let π be an interaction play such that the last position $b = (b, I)$ in π is a* **P***-position. Let I_k be the only* **P***-position in I. Let $s_k = (F, \vec{c})$ where F is neither in $\mathscr{U}_p$ nor in $\mathscr{V}_{i_j}$ for some $j \neq i$. Then there exists an interaction position a, with $a <_\pi b$, where $F \in \mathscr{V}_{i_j}$ for some j.*

Proof. The proof will be by induction on the length of π. If we are at the first move of the play π, then **type 2** cannot be performed. Let c be a position where the condition holds. Let us consider a move $(F, \vec{b})$ from c to d.

O-move from b. If an **O**-move is performed, then $F\vec{b}_*$ is added to $\mathscr{U}_{i_k}$ and to $\mathscr{U}_p$. This means that if one of the $F\vec{b}_*$ are ever picked from $\mathscr{U}_{i_k}$ they will not lead to an i-move because they are in $\mathscr{U}_p$ as well, so that they will lead to a **P**-move.

P-move from b. A **P** move doesn't change any set $\mathscr{U}$, so the condition will still hold after the move.

i-move from b. In a **type 1** i-move, will add formulas both sides, so that the will create a precedent for each new added formula.

In a **type 2** i-move, there is the new added $\mathscr{U}_{i_t} + F\vec{b}_*$ set. For the $F\vec{b}_*$ the freshly added formula create a precedent, while $\mathscr{U}_{i_t}$ is surely a subset of a set $\mathscr{U}$ which is now present c, so the propriety holds. □

Lemma 20 (Well-Definedness and Liveness)**.** *The conditions of Interaction Position are still verified after an interaction move in a interaction play π. Moreover, an interaction move can always be made, except in the case of a* **O***-position where $\mathscr{V}_p = \emptyset$.*

Proof. As mentioned, we have to verify two aspects: that a move can always be made and that the conditions of Interaction Position are still verified after a move.

O-move An **O**-move takes an $F \in \mathcal{V}_{p_i}$. If there is no formula F in $\mathcal{V}_{p_i}$, then the move cannot be executed, and indeed, as we will see later, this corresponds to the winning condition for **P**.

Since p_i is an Interaction Position, we know there exists a set $\mathcal{V}_{i_k}$ that contains F. Clearly the move can also be executed in i_k.

After executing a move, p_{i+1} becomes a **P**-position, and so does i_k. At this point, that k is precisely the k that identifies the unique **P**-position. We now verify if every formula in $\mathcal{U}_{p+1}$ is present in $\mathcal{U}_{i_j}$ for some j. We note that $\mathcal{U}_{p+1} = \mathcal{U}_p + F\vec{b}_*$: the property is clearly verified for every formula in $\mathcal{U}_p$, while the formulas $F\vec{b}_*$ are in $\mathcal{U}_k$. Furthermore, $F\vec{b}_0$ has been added to $\mathcal{A}_p$, which was also added to a $\mathcal{A}_k$, hence the equality $\mathcal{A}_p = \bigcup_{i=1}^{n} A_{i_i}$ continues to hold.

P-move The **P**-move selects an $F \in \mathcal{U}_{p_i}$ which, by definition, is also in a $\mathcal{U}_{i_k}$. Referring to Remark 4, if the move is justifiable in $\mathcal{A}_{i_k}$, then it is justifiable in $\mathcal{A}_p = \bigcup_{i=1}^{n} A_{i_i}$.

After executing a move, p_{i+1} becomes a **O**-position, and so does the unique **P**-position $i_k \in I$ where the move was made. Clearly, the formulas in $\mathcal{V}_p$ are also in $\mathcal{V}_k$. No set $\mathcal{A}$ has had anything added to it.

***i*-move**

(tp. 1) If $F \in V_{i_j}$ for some j, clearly the move can be executed and the resulting position is always a **P**-position. After the move, formulas have been added to $\mathcal{U}_{i_j}$ but not to $\mathcal{U}_p$, while the formula that is added to $\mathcal{A}_{i_j}$ was already present in the union $\bigcup_{i=1}^{n} A_{i_i}$. The position is therefore still an interaction position.

(tp. 2) Thanks to Lemma 19, the move can always be executed. Similarly to (*type 1*), after the move, the position is still an interaction position.

□

Lemma 21. *Let b be position in a play π and let $(\mathcal{U}_k, \mathcal{V}_k, \mathcal{A}_k) \in I_k$. Then, for every formula $F \in \mathcal{U}_k$ which is not in $\mathcal{U}_p$, F is either in some $\mathcal{V}_{i_j}$ with $j \neq k$ or there exists an interaction position a, with $a <_\pi b$, where $F \in \mathcal{V}_{i_j}$ for some j.*

Proof. We do the proof by induction on the length of the play π. If we are in an *initial interaction position* then the Lemma is true by definition of *initial interaction position*. If $(\mathcal{U}_k, \mathcal{V}_k, \mathcal{A}_k)$ is not changed during a move, *i.e.* the move was not involving that specific position, then the propriety is clearly still verified after the move. Let us say that we have a move $(G, \vec{b})$ involving that position: it is a **O** move or an **type 1** i-move.

O-move The effect of the move on that specific position is $(\mathcal{U}_k + G\vec{b}_*, \mathcal{A}_k + G\vec{b}_0)$. For all $F \in \mathcal{U}$ the propriety still holds, while all the $G\vec{b}_*$ are in $\mathcal{U}_p$ as well.

i-move The effect of the move on that specific position is $(\mathcal{U}_k + G\vec{b}_*, \mathcal{A}_k + G\vec{b}_0)$. On another component I_j the effect is $(\mathcal{U}_j, G\vec{b}_*, \mathcal{A}_j)$, so for the new added $G\vec{b}_*$ in $\mathcal{U}_k$, they are in $\mathcal{V}_j$.

□

Definition 27 (Winning Condition)**.** *A play is winning for* **P** *if and only if a* **O***-move cannot be executed because $\mathcal{V}_p$ is empty.*

Definition 28 (Respect a strategy)**.** *A* **P***-move or a i-move are said to respect the strategy if $(F, \vec{b}) = s_k(i_k)$, where i_k is the unique* **P***-position. A play π is said to respect the strategy if every* **P***-move and i-move of π respect the strategy.*

Let us now introduce a graph to better represent an interaction play π. To do this, let us recall the definition of play gave in Chapter 1: a finite or infinite walk in the graph $\mathbf{G_T}$. In Chapter 3, we showed that the subgraph of $\mathbf{G_T}$ where **P**'s moves follow a strategy σ and, among **O**'s moves, only the *generic moves* are considered, has finite branching. In other words, the subgraph is infinite if and only if there exists an infinite walk. Make explicit that the subgraph is restricted to the accessible positions from a given initial position. In this case, unlike the definition of $\mathbf{G_T}$, the graph will not represent a game, but only a specific interaction play. Moreover, the interaction play will respect a strategy.

To simplify the discussion, let us then consider the case where $len(I) = 2$, where I is an *initial interaction position.*

Definition 29 (Interaction Graph of a Play)**.** *$\mathbf{G}_i(\pi)$ is a forest composed of two trees, illustrating the unfolding of an interaction play π that follows a strategy. The play that takes place in the main position p is not shown in the graph. More specifically, the roots of the two trees are the initial positions i_1 and i_2,* **P** *and* **O** *moves are represented with green arrows while the i-moves with black arrows. The relation R_I is represented with dashed arrows.*

For instance, in the Interaction Graph shown in Figure 4.6, the main position p interacts exclusively with the component I_1. The positions in red are the initial positions, and the arrows point in the reverse direction, from the final position of a move back to the starting position. The subscript of each position solely indicates its distance from the root of the tree.

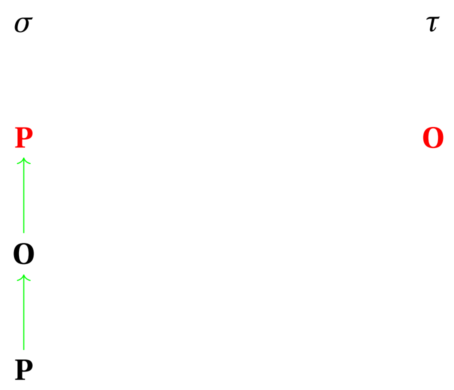


Figure 4.6.: From position **P**, a **P**-move is made to reach position **O**, and then a **O**-move is made to reach position **P**. Meanwhile, the second component I_2 remains in the initial position **O**.

In other words, making a **P**-move or **O**-move means advancing only in the tree with which the main position p is interacting.

Making an i-move, on the other hand, means advancing in both trees, creating an interaction between them. In the example shown in Figure 4.8, the move from $\mathbf{P}_2$ to $\mathbf{O}_3$ is a **type 1** i-move, which moves in both components. The two final positions are connected by a dashed arrow representing the interaction relation R_I, pointing from the **P** position to the **O** position.

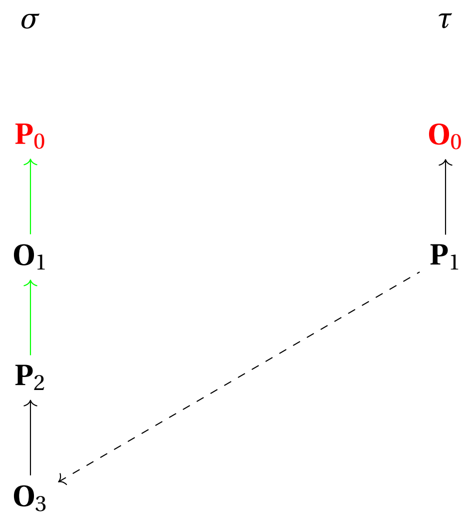


Figure 4.7.: If a position is obtained following an i-move, then an interaction is created, indicated by a dashed arrow going from the **P**-position to the **O**-position.

The situation is slightly different in the case of a **type 2** i-move. For instance, imagine that the next move from $\mathbf{P}_1$ in the tree τ is a **type 2** that interacts with the old position $\mathbf{O}_1$ in the tree σ. At this point a branching is created, as shown in Figure 4.8.

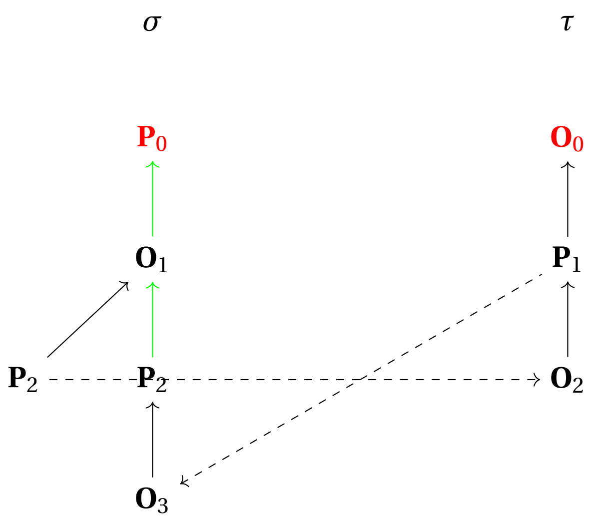


Figure 4.8.: A **type 2** i-move generates a branching.

Should we wish to demonstrate a result similar to that in Chapter 3, we could certainly assume that all **O**-moves are generic moves. However, we could not do the same for all **O**-moves resulting from an i-move.

First, let us focus on the case where only i-moves are played from the beginning (either **type 1** that continue a path or **type 2** that initiate a branching), as shown in Figure 4.9.

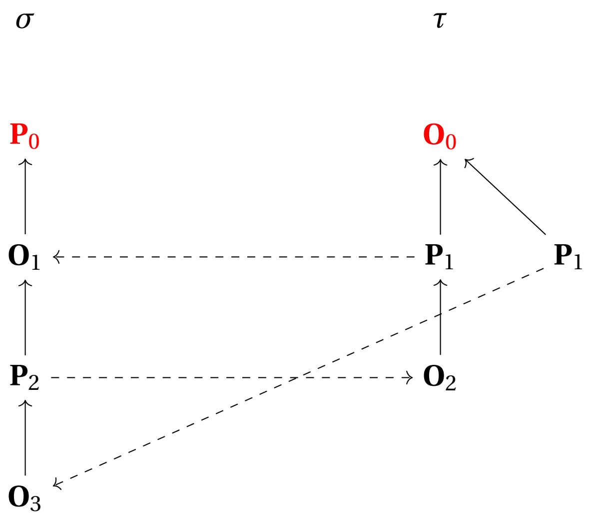


Figure 4.9.: An example of a $\mathbf{G}_i(\pi)$ graph composed solely of i-moves.

It is now useful to observe the following Lemma 22, which limits the possibilities where the type 2 i-move can generate branching. Otherwise, there would be no chance to control the possible interactions that occur between the two trees.

Lemma 22. *If a **type 2** i-move goes from a P_i to an O_{i+1} position in one of the two component, in the other component it goes from an y position to an z position such that it exists a x position in the path that goes from P_i to P_0 such that $R_I(x, y)$.*

Proof. Trivial by looking at Definition 24. □

We now want to show that, given the restriction of Lemma 22, in an infinite play there cannot be an infinite branching in $\mathbf{G}_i(\pi)$.

Conjecture 1. *Let π be a play where only i-moves are performed. Then there does not exist a tree of $\mathbf{G}_i(\pi)$ with an infinite branching.*

Corollary 3. *Let $p = (p, I, R_I)$ be a position in a play π where all s_i are $\mathbf{P}$-winning. If we know a priori the number m of times a **type 2** i-move is played from that position onwards, then $\mathbf{P}$ wins on π.*

Proof. If $len(I) = n$, then there will be maximum $n + m$ sets in each position of π. By absurd, let us assume that the play is infinite and not winning from $\mathbf{P}$. Then it visits an infinite number of time a particular position I_k. Which is absurd because it would mean that s_k is not winning on i_k. □

Corollary 4. *Let p be a position, where all s_i are $\mathbf{P}$-winning. Then there cannot be an infinite sequence of successive **type 2** i-moves starting from position p.*

Conjecture 2. *If $\mathbf{G}_i(\pi)$ is infinite, then one of its trees contains an infinite branch.*

Conjecture 3. *Let p be an Initial Interaction Position, where all s_i are $\mathbf{P}$-winning. Then, $\mathbf{P}$ wins in every possible play that starts from p.*

As we can see, there are still various conjectures that we hope will soon be answered. The idea is that, thanks to Conjecture 3, there is a procedure to always win, but not an actual definition of a strategy σ, composition of strategies s_i.

The next step will therefore be to define the composition operation $\circ$ and, consequently, a strategy $\sigma = (\tau_1, (\mathscr{U}, \mathscr{V}_I, \mathscr{A}_1)) \circ (\tau_2, (\mathscr{V}_I, \mathscr{V}, \mathscr{A}_2))$ in terms of τ_1, τ_2, and two given positions. σ will be winning in a position $p = (\mathscr{U}, \mathscr{V}, \mathscr{A})$ if τ_1 is winning on $(\mathscr{U}, \mathscr{V}_I, \mathscr{A}_1)$ and τ_2 on the position $(\mathscr{V}_I, \mathscr{V}, \mathscr{A}_2)$ for some $\mathscr{V}_I$.

4.3. Further Developments: Full-Abstraction and Common Knowledge

In Chapter 3, the correspondence under appropriate conditions between winning strategies in the game $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ and proofs in $\mathrm{LK_{game}}$ was shown. In this chapter, the composition of strategies corresponding to the elimination of the cut was studied. These two results are sufficient to define full abstraction. It would also be important to explore how to interpret the elimination of the cut in light of the composition of strategies. For example, the **type 2** i-move is likely to correspond to the duplication of proof trees that can occur during S_1 or S_2 cuts.

Furthermore, the composition of strategies provides us with a procedure to enrich the set **T**, which, as we will discuss in the next chapter, could be seen as *Common Knowledge* between the two players: if **P** has a winning strategy on F, then F can be added to the set **T**. More specifically, if there is a winning strategy for $\mathbf{T} \vdash F$ and for $\mathbf{T} + F \vdash G$, then it is known that there is a winning strategy for $\mathbf{T} \vdash G$. However, epistemologically, it is interesting to note that knowing the two strategies is not enough, but one must know how to make them interact correctly: in other words, the two strategies are not read sequentially but in parallel.

5. Real-World Playability and Online Software Implementation

5.1. A Brief Epistemological Framework for Krivine's Normal Form and the Game

In Chapter 1, to represent formulas, we introduced Krivine's normal form

$$\forall \vec{x}\, F_1, \dots, F_n \to A$$

to be read as "for every x, if all the hypotesis $F_1(\vec{x})$, ..., $F_n(\vec{x})$ hold, then the conclusion $A(\vec{x})$ also holds". As shown in Chapter 1, the minimal language $\mathscr{L} = \bot, \to, \forall$ is sufficient to describe all first-order logic up to provability in classical logic. Moreover, any formula written in the minimal language $\mathscr{L}$ can be expressed in Krivine's normal form.

In the next section, we aim to explore whether this normal form—that is, this way of expressing formulas—and the $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ game in general have a meaning that goes beyond their technical usage.

5.1.1. The Statements

In mathematics, we aim to establish general facts, which can help us *predict* (from latin *praedicĕre*, to say beforehand), that is, to know in advance certain behaviors, effects, situations. Knowing that a single object in a set satisfies a certain property is sometimes useful in some contexts, but knowing that all objects in a set satisfy a property holds greater value, because it allows us to make predictions about objects of that kind before practical verification. Although, in the following discussion, we will limit ourselves exclusively to mathematical examples, the reader can seek examples in all branches of scientific knowledge.

Let us start by considering an extremely simple statement: "each element has a square root". In other words, using logical formalism, $\forall x\, \exists y\, (y \times y = x)$. This statement is true in some environments (like complex numbers) and false in others (like integer or real numbers).

More generally, it can be discovered that—among the objects of a set—only some particular objects satisfy a certain property. In other words, one might find that a certain property is true only for those objects in a set that have specific characteristics. For example, in the context of real numbers, it is not true that every number is smaller than its square. But if we limit ourselves to numbers greater than 1, then it becomes true. In logical formalism, even though it's false that $\forall x(x < x^2)$, it is true that $\forall x(x > 1 \to x < x^2)$. As often happens, we note that the hypothesis is a sufficient condition but not a necessary one.

Generally speaking, it's quite intuitive to state a scientific law in the following manner: $\forall x(H_1(x) \wedge H_2(x) \wedge \ldots \wedge H_n(x) \to C(x))$. Put simply, if an object x from a specific set satisfies *all* the conditions H_1, ..., H_n, then it also satisfies the conclusion C. To streamline the notation, we'll replace the conjunction symbol $\wedge$ with a comma, leading to the form $\forall x(H_1(x), \ldots, H_n(x) \to C(x))$.

5.1.2. Debating on a Statement

It's now interesting to see how a statement can undergo critical analysis. For the sake of simplicity, let us imagine a conversation between two individuals with opposing views on the validity of the statement: a dialogical analysis of a statement clearly occurs in any research context, even within a single individual's thought process. How does one argue that a particular statement of the form $\forall x(H_1(x), \ldots, H_n(x) \to C(x))$ is false?

Let us envision a brief conversation between two individuals about a very simple statement.

P: "Did you know that all polygons have at least one obtuse angle?"

O: "That's not true! Look, this triangle has all acute angles."

P: "You're right, but if the polygon has at least 5 sides, then it's true!"

In this conversation, the critical analysis of the statement led to the introduction of a new hypothesis H about the polygon to make the statement true.

It's worth noting that the two interlocutors seem to have a mutual understanding of what constitutes a polygon—for instance, excluding intertwined polygons—and of the definition of an obtuse angle. For any statement to be subjected to analysis, there must be a shared foundational knowledge between the interlocutors. This includes a common language, a set of shared true statements—denoted as $\mathbb{T}$ (which could potentially be empty)—and a set of shared false statements, henceforth referred to as $\perp\!\!\!\perp$. Although $\perp\!\!\!\perp$ can also be empty, it's safe to assume that both players agree on the fact that false is indeed false, *i.e.*, $\perp \in \perp\!\!\!\perp$. Moving forward, it's important to underline that stating $A \to \perp$ is equivalent to stating $\neg A$, meaning claiming that A is false.

Generalizing from the previous example, if one wishes to argue that a statement $\forall x(H_1(x), \ldots, H_n(x) \to C(x))$ is false, they need to present a *counterexample*. Specifically, they must identify a particular object a for which $H_1(a)$, ..., $H_n(a)$ all hold true, and yet, $C(a)$ is false. In more formal terms, one must find a witness for the following formula: $\exists x(H_1(x) \wedge \ldots \wedge H_n(x) \wedge \neg C(x))$.

Let's delve into another example, a slightly more intricate conversation, which will pave the way for generalizing our earlier analysis.

P: "Did you know that all functions defined over a bounded interval have a local maximum?"

O: "No, that's not correct! The function $f(x) = \frac{1}{x}$ doesn't have a maximum in the interval $(0,1)$."

P: "Actually, $f(x) = \frac{1}{x}$ isn't defined on $(0,1)$."

O: "Yes, it is. At which point do you think it's undefined?"

P: "At 0!"

O: "Look, 0 isn't part of the interval $(0,1)$."

P: "Right..."

The same conversation, expressed in logical formalism, goes as follows:

P: "$\forall f \forall I(H_1(f,I), H_2(I) \to C(f,I))$"

Where H_1 stands for the condition that f is defined over I, H_2 indicates that the interval is bounded, and C means that f has a local maximum within the interval I.

O: "$(\forall f \forall I(H_1(f,I), H_2(I) \to C(f,I))) \to \perp$. In fact, $H_1(\frac{1}{x},(0,1)) \wedge H_2((0,1)) \wedge \neg C(\frac{1}{x},(0,1))$"

P: "$H_1(\frac{1}{x},(0,1)) \to \perp$. Indeed $\frac{1}{0}$ is not defined."

O: "$0 \notin (0,1)$"

P reflects and returns to **O** with a statement refined with a new hypothesis.

P: "Did you know that all functions defined over a closed and bounded interval have a local maximum?"

O: "That's still not right! Consider the parabola $f(x) = 1 - x^2$ and, at its vertex, redefine the function to be 0."

P: "The parabola you're describing isn't defined over a closed interval!"

O: "Yes, it is. Just define it over the interval $[-1,1]$."

P: "You're right..."

The revised statement introduced by **P** is $\forall f \forall I(H_1(f,I), H_2(I), H_3(I) \to C(f,I))$, where H_3 represents the property of the interval being closed.

P reflects further and returns to **O** with an even more refined statement.

P: "Did you know that all continuous functions defined over a closed and bounded interval have a local maximum?"

The statement is $\forall f \forall I(H_1(f,I), H_2(I), H_3(I), H_4(f,I) \to C(f,I))$ where H_4 express the propriety of a function being continuous on a domain.

We now observe that, in the preceding example, the hypotheses H may have a layered nature. Generally speaking, the hypotheses H can mirror the form of a statement seen earlier: $H = \forall y(G_1(y) \wedge \ldots \wedge G_m(y) \to G_0(y))$. By adopting this approach, we can expand upon the dialogue rules previously discussed: once **O** asserts that, with a certain witness t, $H_1(t) \wedge \ldots \wedge H_n(t)$ holds true, **P** can counter by claiming that a specific $H_i(t) = \forall y(G_1(y) \wedge \ldots \wedge G_m(y) \to G_0(y)$ is actually false. That is to say, for a particular u, while $G_1(u) \ldots G_n(u)$ all hold true, $G_0(u)$ turns out to be false, thereby continuing the discussion.

In our presented case, the conclusion C also has a layered nature. However, as presented in Chapter 1, one can always rewrite the formula to ensure that C is of a simpler nature, in a sense directly verifiable, shifting all the complexity to the hypotheses.

This kind of dynamic discussion seems to aptly simulate the process of scientific discovery. One might initially conjecture $\forall x C(x)$. However, upon observation and contemplation, it's revealed that not all x satisfy the property C, but only those for which the hypotheses H_1, ... H_n hold, leading to the formulation of a theorem. It is interesting to note that the word "theorem" originates from the Greek *theorema*, meaning 'contemplation', which in turn derives from *theoréo*, 'I see', 'I observe'.

In concluding this section, we emphasize that the scientific process and progress have three foundational characteristics: firstly, creating new hypotheses to be tested out; secondly, testing a hypothesis by challenging oneself, others, or reality through experiments; and lastly, developing theories to categorize and interpret the knowledge. Not only does the game $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ provide an environment to test one's hypotheses, but the set $\mathbf{T}$ captures the idea of adding new facts to shared knowledge, as already discussed at the beginning of chapter 2.

5.2. Online Implementation: the λuì Software

In this section, we present an educational transposition of the game $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$, exploring its practical usability in a learning context. As we will see at the end of the chapter, the software could also be valuable as a proof-search program.

In presenting the software, we consider mathematical environments that on one hand are relevant in educational practice, and on the other hold distinct logical value, even from a historical perspective. Besides "pure" contexts of propositional logic and first-order logic, the online implementation allows playing also in the environments of natural numbers, mathematical analysis, and Euclidean geometry. Specifically, PA (Peano Arithmetic) is of fundamental importance in logic for various reasons including the incompleteness theorems, and for obvious reasons in education. In this regard, we note that PA, with the successor function, reflects the intuitive idea of a numerical system that is built up gradually by adding ever larger numbers. Moreover, it is notably the first area where students encounter nested quantifiers and serves as an environment for studying constructivism. Euclidean geometry (where it is still part of the curriculum!) serves as a fundamental setting for learning the logical structure of statements, with particular emphasis on implication. Historically, it has epitomized logical rigor within mathematics, to such an extent that, in 1821, A. L. Cauchy began his *Cours d'Analyse de l'École Royale Polytechnique* by stating his intention to endow the methods of analysis with "all the rigor that is demanded in geometry". Furthermore, in any logic course that discuss axioms, Euclid's axioms are invariably explored.

In other words, this chapter aims to serve as a bridge between the first four chapters, which are exclusively logical in their nature, and the last four, which are predominantly related to Mathematics Education.

Let us now delve into the online implementation of the $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ game, called λ**ui**. The game was developed with the essential contributions of Mattia Sanchioni (for managing the code logic and generally the backend) and Luca di Pietro Martinelli (for managing the website where the code is run and generally the frontend, handling UI and UX).

The game features two players challenging each other regarding the truth of F, a

Figure 5.1.: Homepage of λui.

formula written in Krivine's normal form. The *Proponent*, abbreviated as **P**, believes F to be true, while the Opponent, abbreviated as **O**, believes it to be false. The game follows dialogue rules that formalize the insights given in the previous section.

The software is split into a backend, accessible at www.galua.cc, which manages the game's logic and data storage, and a frontend available at www.oiler.education/lui, where the game is intended to be played, which exclusively handles the visualization of the game's dynamics[1]. The name "Luì" is inspired by the French pronunciation of Jean-Louis Krivine, the designer of $\mathscr{U}\mathscr{V}\mathscr{A}$ game (Krivine and Legrandgérard 2007).

At www.oiler.education/lui, users can find and freely access the software. Here, they have the choice of playing in either propositional logic or first-order logic (Figure 5.1). Within the domain of first-order logic, there are four available theories to select from: **pure logic** (where no specificy theory **T** is involved), **Giuseppe Peano** (PA arithmetic), **Auoquamel** (analysis), and **Alfred Tarski** (Euclidian geometry). As we will later discuss, adjustments have been made on these theories to ensure their playability.

Once users have picked their desired theory, they proceed to select a formula of that theory and start the game by clicking on 'START'. As can be seen from Figure 5.1, a 'User vs PC' mode is also planned for the future. For now, the game is played between two real players (*Proponent* and *Opponent*) who play on the same device.

During the game, the three sets $\mathscr{U}$, $\mathscr{V}$, and $\mathscr{A}$ are referred to as $\mathbf{O}_\top$, $\mathbf{P}_\top$, and $\perp\!\!\!\perp$ respectively, for easier comprehension on their status. Indeed, $\mathbf{O}_\top$ represents what is true for the Opponent, $\mathbf{P}_\top$ what is true for the Proponent, while $\perp\!\!\!\perp$ represents what is false for both players. When a theory is present, it is denoted by $\top\!\!\!\top$, underlining its symmetric and opposite relationship with $\perp\!\!\!\perp$.

In fact, there is a strong duality between the sets $\top\!\!\!\top$ and $\perp\!\!\!\perp$: the former contains

[1]The frontend and backend communicate via REST API: the frontend initiates a call to an endpoint on www.galua.cc, passing all required parameters, and the server responds with the requested data.

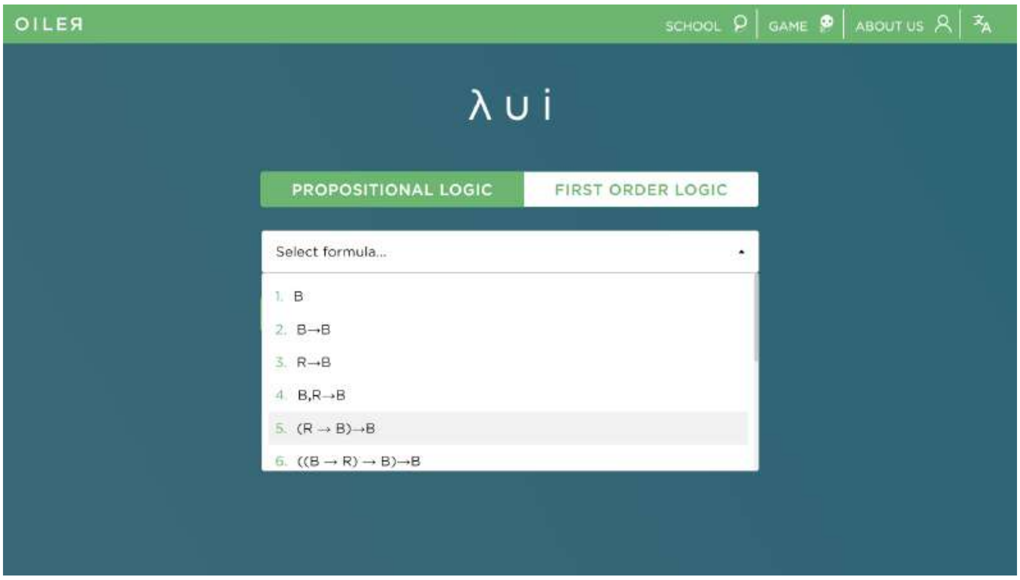


Figure 5.2.: In Propositional Logic, users have a choice among 13 distinct formulas.

formulas that are true for both players, and the latter contains formulas that are false for both players. The union of these two sets constitutes what can be referred to as *common knowledge*. However, as we will see in more detail when presenting the rules of the game, this shared knowledge has a peculiarity, and the dualism between $\top\!\!\top$ and $\bot\!\!\bot$ becomes even more evident: the formulas in the set $\top\!\!\top$ can only be invoked by **P** (we might say removed from $\top\!\!\top$, barring contractions), while **O** is the only one who can add formulas to $\bot\!\!\bot$. The $\top\!\!\top$ remains constant during the game, whereas the $\bot\!\!\bot$ is not. Indeed, this asymmetry makes sense: the Opponent, in their attempt to deny the formula F, has no interest in making concessions of truths. Symmetrically, the Proponent has no interest in conceding false formulas.

To facilitate reading, if during the game the user hover the cursor over a formula without clicking it, the hypotheses of a formula are marked in orange, while the conclusion is in dark blue. The top-level quantifiers $\forall$ and the main implication $\rightarrow$ are in black. As an example, a formula is written as $\forall x(F_1(x), \ldots, F_n(x) \rightarrow A(x))$.

For every game modality (*i.e.*, every theory), we will provide the specific rules of that modality, evidently adapted from the general rules outlined in Chapter 1.

5.2.1. Propositional Logic

The game is played between **P** and **O** on a propositional formula F; in Propositional Logic, users have a choice among 13 distinct formulas (Figure 5.2). The formulas aim to provide a progressive and meaningful approach to propositional logic.

The game initializes with $\mathbf{O}_\top = \{F \rightarrow \bot\}$ (*i.e.*, **O** believes F to be false), $\mathbf{P}_\top = \{F\}$ (*i.e.*, **P** believes F to be true), and $\bot\!\!\bot = \{\bot\}$ (*i.e.*, both players concede that $\bot$ is false). The game starts with **O** playing first, and the turns alternate thereafter.

- **O** plays by choosing a formula $F \in \mathbf{P}_\top$. They add all the premises $F_1, \ldots, F_n$ of F

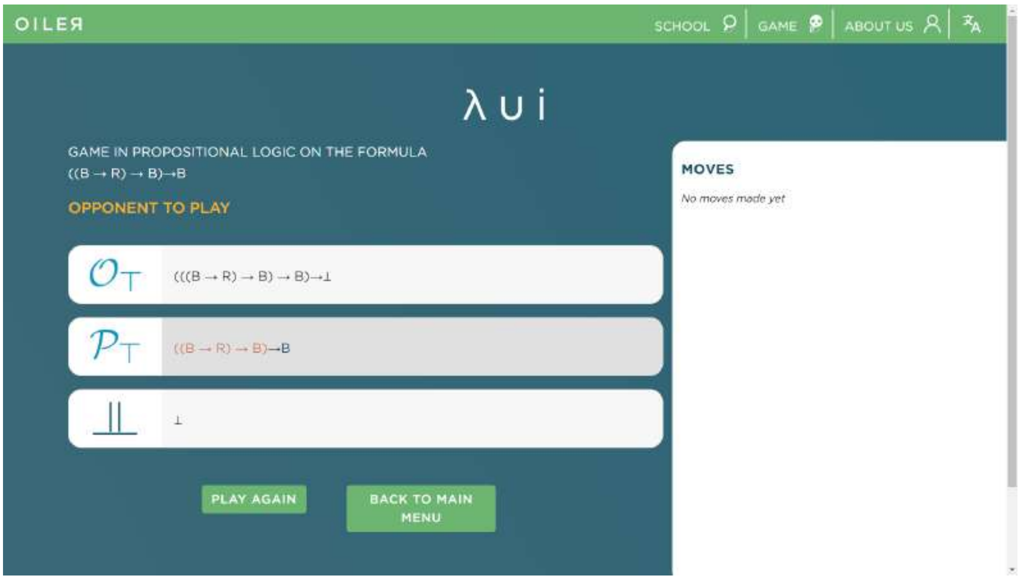


Figure 5.3.: The Opponent's opening move, where they pick $((B \to R) \to B) \to B$.

to $\mathbf{O}_\top$ and the conclusion F_0 to $\perp\!\!\!\perp$. In particular, if $\mathbf{P}_\top$ is empty then $\mathbf{O}$ cannot move.

- $\mathbf{P}$ plays by choosing a formula $F \in \mathbf{O}_\top$ such that the conclusion F_0 is in $\perp\!\!\!\perp$. They replace the set $\mathbf{P}_\top$ with $\{F_1, \ldots, F_n\}$.

If the play is finite, which is equivalent to say that **O** can no longer make a move, then **P** is the winner; otherwise, **O** wins. Clearly, since **O** wins if and only if the play is infinite, the message "Opponent wins" does not exist.

As an example, let us analyze a match on Pierce's formula $((B \to R) \to B) \to B$.

The game begins with the Opponent's move (see 5.3), where they select the only formula they can from the set $\mathbf{P}_\top$, namely $((B \to R) \to B) \to B$. They assert that B is false, placing it into $\perp\!\!\!\perp$, and that $((B \to R) \to B)$ is true, moving it to $\mathbf{O}_\top$.

The Proponent now has two options (see 5.4): either restart the game by choosing $(((B \to R) \to B) \to B) \to \perp$ (which is the negation of Pierce's formula), or selecting $((B \to R) \to B)$. The move is valid because $B \in \perp\!\!\!\perp$. Clearly, continuously restarting the game in the long run isn't a favorable strategy, so **P** opts for $((B \to R) \to B)$.

The Opponent, still following a predetermined path, proceeds by adding $R \in \perp\!\!\!\perp$ and $B \in \mathbf{O}_{\top\!\!\top}$ (see 5.5).

The Proponent wins since B belongs to both $\mathbf{O}_\top$ and $\perp\!\!\!\perp$ (see 5.6 and 5.7).

5.2.2. First Order Logic: Pure Logic

The game is played between **P** and **O** on a first-order formula F. In Pure Logic users have a choice among 13 distinct formulas, as shown in Figure 5.8. Similarly to Propositional Logic environment, the formulas strive to follow a progressive development of skills: starting from very simple formulas, moving through the Drinker Paradox

Figure 5.4.: Proponent's first move.

(formula number 7), to binary predicates[2]. The game initializes with $\mathbf{O}_\top = \{F \to \bot\}$, $\mathbf{P}_\top = \{F\}$, and $\perp\!\!\!\perp = \{\bot\}$. The game begins with **O** playing first, and the turns alternate.

- **O** plays by choosing a formula $F \in \mathbf{P}_\top$ and closed terms $\vec{b}$ to be substituted to variables $\vec{x}$ of the top-level quantifiers of F. They add $F(\vec{b})_1, \ldots, F(\vec{b})_n$ to $\mathbf{O}_\top$ and $F(\vec{b})_0$ to $\perp\!\!\!\perp$. In particular, if $\mathbf{P}_\top$ is empty then **O** cannot move.
- **P** plays by choosing a formula $F \in \mathbf{O}_\top$ and closed terms $\vec{b}$ such that $F(\vec{b})_0 \in \perp\!\!\!\perp$. They replace the set $\mathbf{P}_\top$ with $\{F(\vec{b})_1, \ldots, F(\vec{b})_n\}$.

We note that a closed term—in the Pure Logic mode—is simply a letter (*e.g.*, a, b, ...); at each turn, the player can choose whether to play a letter that has been played previously in the game or a new letter, referred to as a *fresh constant*[3]. If the play is finite, which is to say that **O** can no longer make a move, then **P** is the winner; otherwise, **O** wins. Clearly, in this instance as well, since **O** wins if and only if the play is infinite, the message "Opponent wins" does not exists.

Let us see a play on the formula number 5, which is $\forall x((\forall y(G(y) \to \bot) \to \bot) \to G(x))$. On the right, we can see the set of moves: each move is presented as $(F, \vec{b})$ where F is the picked formula and $\vec{b}$ are the picked closed terms.

The first to move is **O**. They select the only formula in $\mathbf{P}_\top$ and choose the closed terms to replace x. Since no closed term has been played yet, the only move they can make is to play a fresh constant, namely the first letter a.

[2]For completeness, we inform the reader that the formula $\forall x(\forall y(A(x,y) \to \bot), (\forall z(\forall w A(w,z) \to \bot) \to \bot) \to \bot)$ corresponds to $\exists y \forall x A(x,y) \to \forall x \exists y A(x,y)$ while the formula $\forall x(\forall y A(y,x) \to \bot), \forall z(\forall w(A(z,w)\bot) \to \bot) \to \bot)$ correspond to $\forall x \exists y A(x,y) \to \exists y \forall x A(x,y)$. Clearly, the first one is false and the second is true.

[3]In other words, during their move, the two players can choose the witness from among those played up to that point, or a new one.

Figure 5.5.: Opponent's move.

The conclusion $G(a)$ is subsequently added to $\perp\!\!\!\perp$, and the only premise is added to $\mathbf{O}_\top$. It's now to **P** to play. They have the choice of either playing $F \to \perp$ (restarting the game) or playing the formula, just added by $\mathbf{O}_\top$, $\forall y(G(y) \to \perp) \to \perp$: they're allowed to make this move since $\perp \in \perp\!\!\!\perp$.

O is now obligated to select the only formula present in $\mathbf{P}_\top$. However, they do have the choice to replace y with either a previously played closed term (namely a) or introduce a new one. This decision is crucial.

If **O** makes the unfortunate choice of playing a, then **P** will win in the subsequent turn. On the other hand, by choosing a new letter (and doing so every time the opportunity arises), the Opponent manages to perpetually continue the game, thus winning. It's worth noting here—as extensively discussed earlier—that the formula is false because a winning strategy exists for the Opponent. However, if the Opponent makes erroneous choices, they can still lose.

5.2.3. Giuseppe Peano

5.2.3.1. PA Theory

Peano's theory is an axiomatic system that aims to describe the set of natural numbers with elementary operations. It was introduced by the Italian mathematician Giuseppe Peano in 1889. The updated first-order theory is today referred to as Peano Arithmetic, or more simply PA. The language used in PA includes symbols for functions $0, s, +$, and $\times$ where s denotes the function that assigns the successor to every natural number. The only relation symbol is $=$. The axioms, in addition to the three standard ones for equality[4], are as follows:

[4]Hereafter, we will refer to these axioms as EQ1 (reflexivity), EQ2 (symmetry), and EQ3 (transitivity).

Figure 5.6.: Proponent wins by picking B from $\mathbf{O}_\top$.

- (PA1) $\forall x \neg(s(x) = 0)$
- (PA2) $\forall x \forall y (s(x) = s(y) \rightarrow x = y)$
- (PA3) $\forall x (x + 0 = x)$
- (PA4) $\forall x \forall y (x + s(y) = s(x + y))$
- (PA5) $\forall x (x \times 0 = 0)$
- (PA6) $\forall x \forall y (x \times s(y) = (x \times y) + x)$
- (PA7) the axiom schema for the Induction Principle

5.2.3.2. On the theory used in the game

First of all, let us notice that the theory we just introduced has closed terms: consequently, it would be possible to effectively play it in a $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ game. However, doing so would be exceedingly cumbersome and tedious, even for a logic enthusiast. If we hope to achieve a truly playable game, one requirement we cannot avoid is that the two players should be able to directly input, via keyboard, natural numbers as closed terms. This requirement makes the function s superfluous, as it can be emulated by the unary function $x + 1$[5]. The need to enter natural numbers brings with it the introduction of another axiom schema: for each n, we have $\overline{n} + 1 = \overline{n+1}$, where n is a constant symbol, and $\overline{n}$ is the interpretation of the constant in the model. However, if a player were to justify the status of the number n based on these axioms every time n it's played, the game would still be overwhelmingly tedious.

[5]Consequently, the axioms PA1 and PA2 are rewritten using the function $+$.

Figure 5.7.: Victory screen for **P**.

In the same fashion, justifying every single operation based on the axioms of addition and multiplication would be exceedingly long. For this reason, we decided to let the program handle every operation and the status of the natural numbers by itself. Specifically, every time a closed term appears in the game, it is replaced with the corresponding natural number. In other words, by doing so, the program autonomously manages the axioms PA3, PA4, PA5, PA6, which can thus be dropped from $\mathbb{T}$.

We have also added, as symbols for predicates, the usual predicates in elementary arithmetic practice: $<$, $>$, `EVEN`, `ODD`, `DIVIDE`, `PRIME`. Every predicate P thus added, which we will call a *derived* predicate, is inserted into the theory $\mathbb{T}$ with the axiom

$$P(x) \iff F_P(x)$$

where $F_P(x)$ is a formula written in the language without P. We note that clearly `PRIME` depends on `DIVIDE`: before being able to define `PRIME` as a derived predicate, it is advisable to enrich the language with the symbol for the `DIVIDE` predicate. Since the connective $\iff$ is not part of the language, for each predicate P, formulas P_1 and P_2 have been added to the axioms $\mathbb{T}$, one for each side of the implication.

The symbol for the `DIVIDE` predicate is expressed as $\lhd$ even though standard mathematical practice uses the symbol $|$. This was done because the symbol $\lhd$ captures much more effectively than $|$ the fact that the binary predicate `DIVIDE` induces an order relation over $\mathbb{N}$. We believe that the order relation thus induced is an order relation that is worth delving into at an educational level, for two distinct reasons: firstly, unlike the classic $<$, it is a non total order relation and, additionally, it admits both a minimum, which is 1, and a maximum, which is 0. This is of interest in the definition of the least common multiple: in elementary definitions, "least" refers to the $<$ order; it is therefore necessary to exclude 0 in the definition, limiting oneself to

Figure 5.8.: In Pure Logic (FOL), users have a choice among 13 distinct formulas.

positive numbers.

5.2.3.3. The Game

In the GIUSEPPE PEANO game mode, the user chooses which formula to play from 26 options and whether to play in *SHORTCUT* or *STANDARD MODE* (see Figure 5.13), a distinction that will be elaborated upon later.

The 26 formulas aim to capture some important aspects of number theory. The first three formulas, one true and two false, are used exclusively to familiarize players with the dynamics of the game. Subsequently, simple yet fundamental situations involving even and odd numbers—often overlooked in traditional teaching—are addressed. Following this, Fermat's Last Theorem is proposed, limited to the cases $n = 2$ (*i.e.*, the search for Pythagorean triples) and $n = 3$, where the search for triples can be intriguing and stimulating, although fruitless as demonstrated by L. Euler.

Formula 12[6] is connected to the search for fractions that approximate the square root of 2. Indeed, the term $\frac{1}{x^2}$ becomes increasingly irrelevant as x and y grow. The possible approximation has been known since ancient times (Maracchia 2005), and the numbers xs that satisfy the relation are called *lateral numbers*, while the ys are *diagonal numbers*. It should be noted that to find all lateral and diagonal numbers, one should also study the formula $\forall x, y(2 \times x^2 = y^2 + 1 \rightarrow \perp)$.

Formulas 13, 14, and 15 allow for an in-depth exploration of divisibility and, more generally, proofs in PA. For example, with formula 13, although **P** can always win easily, it's not immediately obvious why this is the case. From 16 to 20, the focus is exclusively on the definition of the PRIME predicate, a foundational concept in number theory. From 21 to 26, typical number theory formulas involving primality are proposed. It

[6] $\forall x, y(2 \times x^2 + 1 = y^2 \rightarrow \perp)$

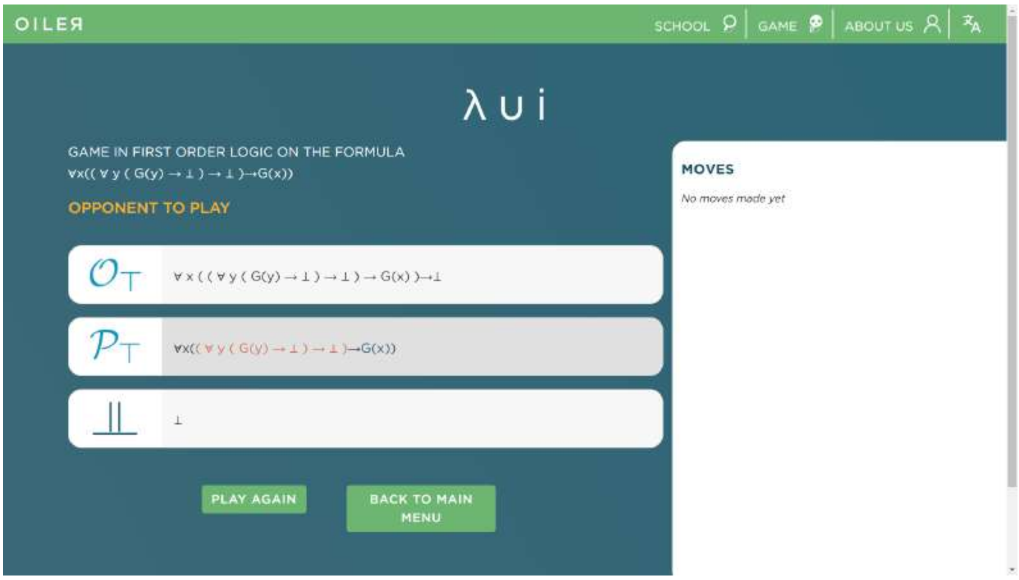


Figure 5.9.: Initial position: **O** plays the only available formula in $\mathbf{P}_\top$.

was decided to conclude with the Goldbach Conjecture[7], as easy to understand as it is difficult to prove. In fact, the conjecture as stated is *false*: we did not specify the condition that the even number must be greater than 2. It will be interesting to see if the Opponent can win by exploiting this gap.

As already mentioned, when players choose to play in GIUSEPPE PEANO mode, they can decide whether to play in *STANDARD MODE* or *SHORTCUT MODE*.

STANDARD MODE

The game is played between **P** and **O** using a first-order formula F written in PA. The game is initialized as usual.

- **O** plays by choosing a formula $F \in \mathbf{P}_\top$ and natural numbers $\vec{n}$ to be substituted to variables $\vec{x}$ of the top-level quantifiers of F. They add $F(\vec{n})_1, \dots, F(\vec{n})_n$ to $\mathbf{O}_\top$ and $F(\vec{n})_0$ to $\perp\!\!\!\perp$. In particular, if $\mathbf{P}_\top$ is empty then **O** cannot move.

- **P** plays by choosing a formula $F \in \mathbf{O}_\top$ and natural numbers $\vec{n}$ such that $F(\vec{n})_0 \in \perp\!\!\!\perp$. They replace the set $\mathbf{P}\top$ with $\{F(\vec{b})_1, \dots, F(\vec{b})_n\}$.

This mode, despite its theoretical interest, is still too complex for practical play, at least initially. Therefore, in addition to all the modifications already implemented to facilitate the game, an additional rule is added in SHORTCUT mode to further simplify it.

SHORTCUT MODE

In the SHORTCUT mode, axioms PA1, PA2, EQ1, EQ2, and EQ3 are removed from the theory $\top\!\!\!\top$. However, a new rule for **P** is introduced: every time a true equality appears in $\perp\!\!\!\perp$, the Proponent has the right to click on it, winning the match. Similarly,

[7] Which, in fact, is attributed to L. Euler!

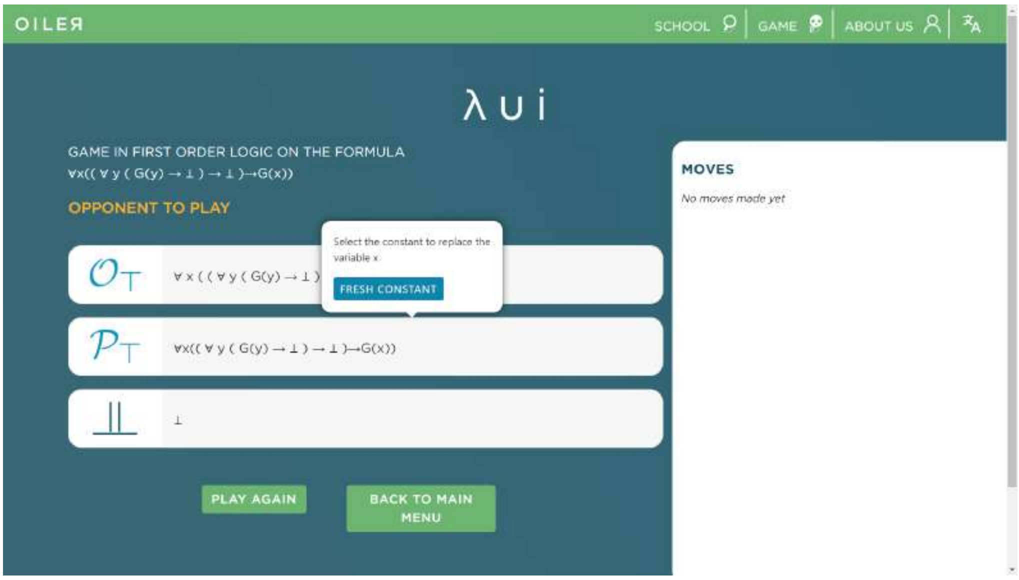


Figure 5.10.: **O** selects the only available formula and replace x with a fresh constant.

whenever a formula of the kind $\forall\vec{x}\ (F_1,\ldots,F_n \to t_1(\vec{x}) = t_2(\vec{x}))$ appears in $\mathbf{O}_{\top}$, the Proponent can always play it, provided that—after the substitution of variables with closed terms—$t_1 = t_2$ is false. This approach is virtually the same to including all true equalities into ⫪ and all the false ones into ⫫. This concept also echoed in Krivine's work (Krivine 2006). In other words, **P** and **O** leave the burden and honor of evaluating equalities to the computer.

More generally, considering potential future developments, the SHORTCUT rule can be implemented in reference to any predicate P, leaving it to the computer to evaluate the predicate. This is done by virtually inserting all $P(\vec{(n)})$ for which P is false into ⫫, and all $P(\vec{(n)})$ for which P is true into ⫪. In a sense, once one becomes accustomed to handling a certain predicate, a method for evaluating its truthfulness is also shared, without having to justify it each time up to basic definitions. Similarly, once the winning strategy for **P** on a formula F is found, this can be inserted into **T**, emulating the evolution of shared knowledge between **P** and **O**

Let us consider an example, in SHORTCUT mode, of a game on the formula PRIME(221). The game is initialized with **P** asserting that 221 is prime, and **O** asserting that it is not prime, as shown in Figure 5.14.

In the first move of the game, Player **O** places PRIME(221) in ⫫, see Figure 5.15.

At this point, **P** recalls the definition of a prime number, stating that if **O** claims that it is not true that 221 is prime, **O** must provide a number that divides 221 that is different from both 1 and 221 (Figure 5.16).

Now, **O** could lose if they provided a wrong witness, such as a number that does not divide 221. However, since 221 is not prime, there are correct witnesses, such as 13. At this point, Player **O** claims that 13 divides 221, but that 13 is neither equal to 1 nor to 221 (see Figure 5.17).

The game could continue with **P** recalling the definition of the `DIVIDE` predicate, to

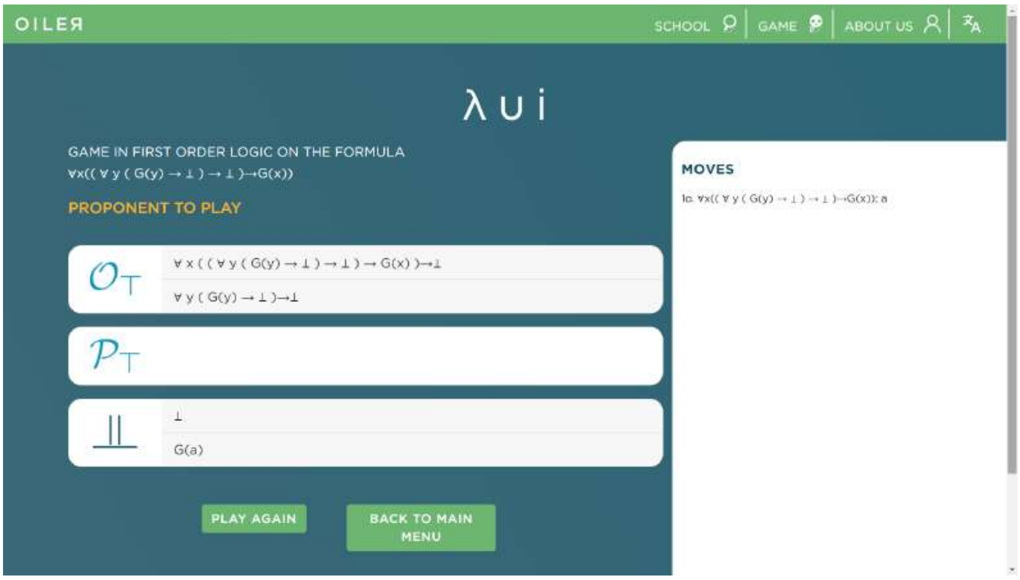


Figure 5.11.: **P** plays $\forall y(G(y) \to \bot) \to \bot$.

invite **O** to find a number k such that $13 \times k = 221$. Clearly, here too, **O** is able to identify the correct witness k. In the end, **P** can do nothing but repeat the same moves, and the game will result in an infinite loop. In other words, **O** will never fall into contradiction.

We conclude the section with a lemma that ensures the two game modes presented are equivalent.

Lemma 23. *The SHORTCUT MODE and the STANDARD MODE are equivalent up to winning strategy for* **P**.

Proof. The SHORTCUT MODE introduces two simplifications: first, whenever a true equality appears in $\perp\!\!\!\perp$, **P** can point it out and win the game; second, to play a formula $F = \forall \vec{x}\,(F_1, \ldots, F_n \to t_1(\vec{x}) = t_2(\vec{x}))$ contained in $\mathbf{O}_\top$, it is sufficient that, after the instantiation of the variables $\vec{x}$, $t_1 = t_2$ turns out to be false, regardless of whether it belongs to $\perp\!\!\!\perp$ or not.

Regarding the first simplification, since sums and products are managed by the computer in both modes, it is enough to note that any true equality is of the type $n = n$ for some natural number n. Therefore, if the equality $n = n$ appears in $\perp\!\!\!\perp$, the game is easily won in both modes: in the SHORTCUT MODE, it suffices to click on the equality in question, while in the STANDARD MODE, it is sufficient to play the axiom $\mathrm{EQ1} = \forall x(x = x)$, choosing the constant n.

Regarding the second simplification, we need to show that—in the STANDARD MODE—it is always possible for **P** to make **O** admit a false equality. To do this, note that, since sums and products are managed by the computer, a false equality is always of the type $n = m$ with n and m natural numbers different from each other. Furthermore, thanks to the axiom EQ2 (symmetry of equality), we can always assume $n < m$. Thus, to make a false equality of one's choice appear in $\perp\!\!\!\perp$, it is sufficient to first play

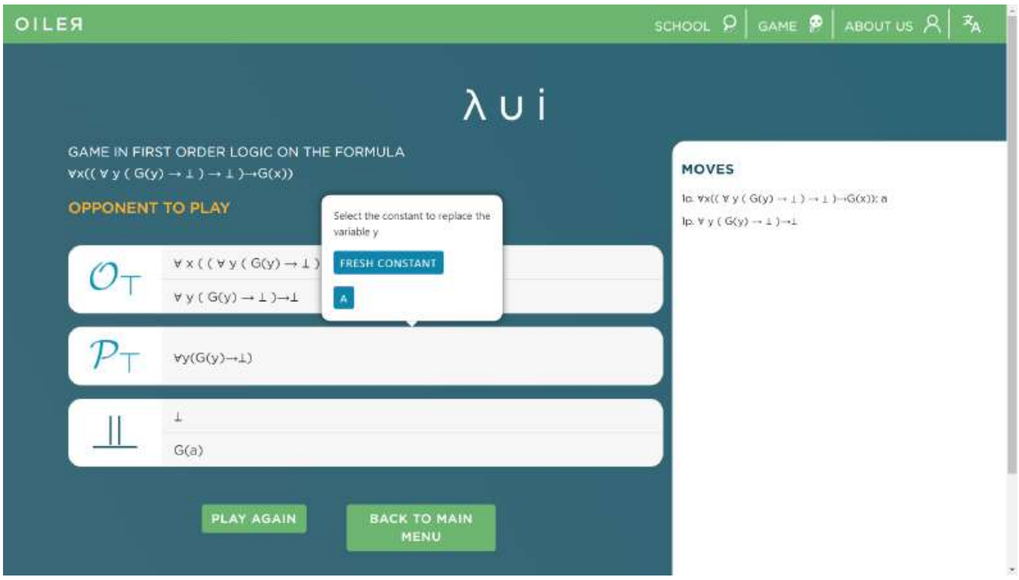


Figure 5.12.: **O** decides on the closed term.

the axiom PA1 with $k = m - n$ and then repeatedly play the axiom PA2 until the desired equality is obtained.

As can be noted, throughout the proof, the axiom EQ3 (not present in the SHORTCUT MODE) was not useful in proving the equivalence of the two modes. Indeed, EQ3 is not useful even in the STANDARD MODE and could be safely removed from $\mathbb{T}$ like the axioms PA3, PA4, PA5, and PA6. □

5.2.3.4. Differences with the Formal Game: On the Induction Principle and True Formulas in $\mathbb{N}$

The version of PA we've defined doesn't precisely mirror the theoretical game for two distinct reasons: firstly, it's not true that infinitely many constants are fresh for $\mathbb{T}$; and secondly the Induction Principle is not included in the theory $\mathbb{T}$.

Concerning the first discrepancy, players can only use numbers in the game, each described by the theory. This means that winning strategies in λuì may not correspond to proofs in PA. When discussing a game on formula F, there is a possibility that while having *for each possible play a winning strategy*, one might not have *a unique winning strategy for every possible play.*

However, we can agree that the condition "for each possible play there is a winning strategy" is a necessary condition for "there exists a winning strategy for every possible play". Once a student has for every play a winning strategy, they can be encouraged to generalize the reasoning, explaining why they are sure to win, no matter which numbers **O** will play. Specifically, by using variables in the Opponent's moves instead of constants.

Relating to the second reason why λuì doesn't perfectly mirror the theoretical game, it's worth noting that the Induction Principle (IP) is not included in $\mathbb{T}$. This does not

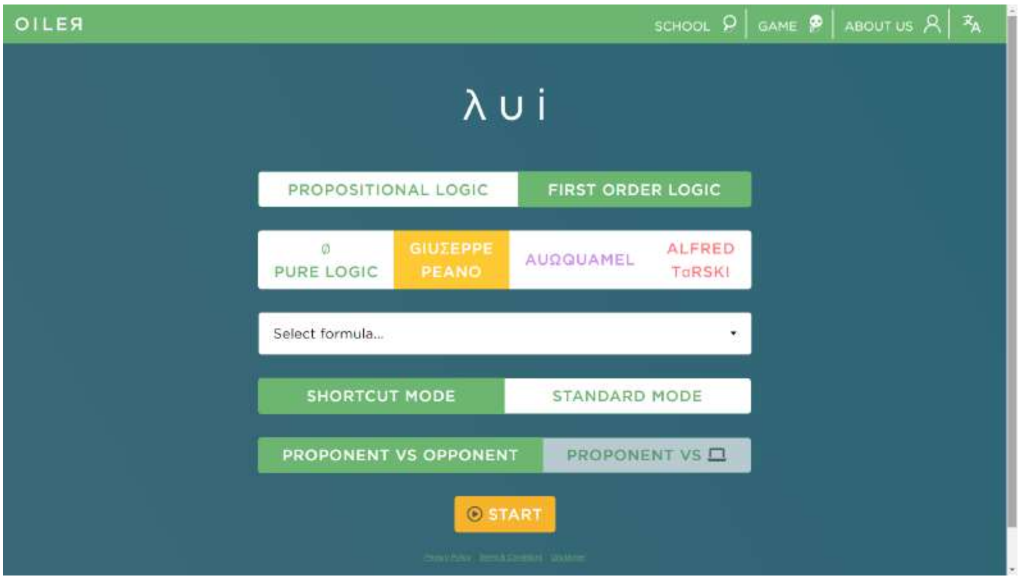


Figure 5.13.: Users set up their game within the GIUSEPPE PEANO mode.

change the potential of having a winning strategy since IP is valid in $\mathbb{N}$. What changes is that the Induction Principle provides the ability to generalize reasoning. Without it, winning strategies might not correspond to derivations. IP can indeed be presented from this perspective, namely as a tool for generalizing reasoning. We plan to add the possibility to play with the IP soon.

5.2.4. Auoquamel

5.2.4.1. Real Numbers

There are various axiomatizations for real numbers, and we find it interesting to mention two qualitatively different approaches here. In 1936, Tarski proposed an elegant second-order axiomatization, which allows for the discussion of completeness (*i.e.*, every non-empty set that is bounded above has a least upper bound), but is clearly unsuitable for our purposes due to its second-order nature. It's worth noting that, in any case, any course in mathematical analysis implicitly considers a second-order structure. On the other hand—as far as first-order is concerned—the theory of *real closed fields* is usually considered. This theory has as models all those that are elementarily equivalent to the real numbers using the standard language (*i.e.*, those models that satisfy all and only the first-order formulas satisfied by the real numbers).

More specifically, a real closed field F is a totally ordered field where every positive element of F has a square root in F, and any polynomial of odd degree with coefficients in F has at least one root in F. This theory was proven to be decidable by A. Tarski.

An example of a model of this theory is the set of algebraic numbers, which could theoretically be utilized in the $\mathbf{T}\mathscr{UVA}$ game. In this game, playing a constant means selecting a specific polynomial and specifying (with the order relation) which root

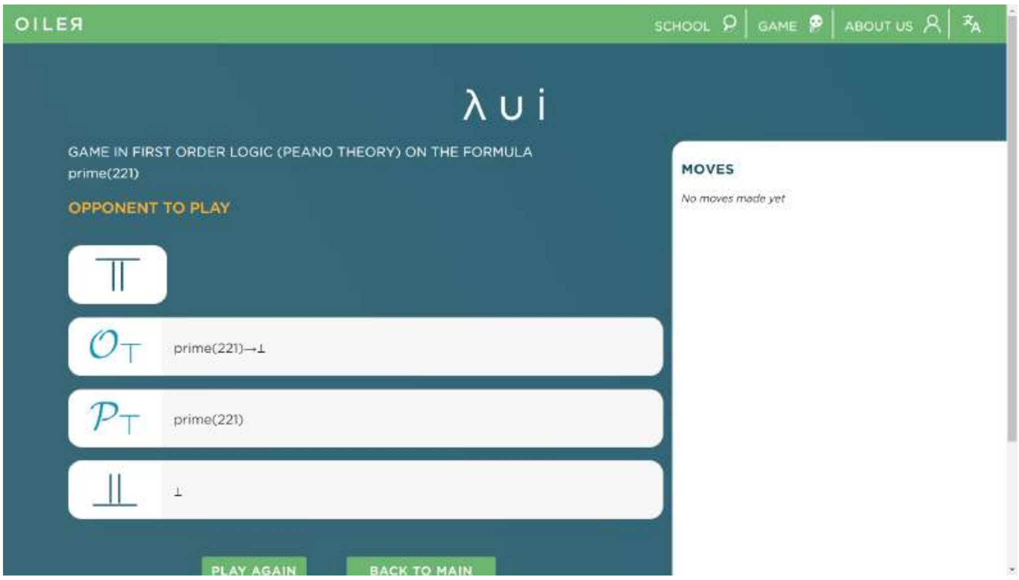


Figure 5.14.: The user sets their game within the GIUSEPPE PEANO mode.

to consider. However, despite being theoretically feasible, this approach would be impractical and artificial in a real-world setting. Additionally, it would be unsatisfactory as it would not allow for the play of commonly used constants like π or e. From a theoretical perspective, the real closed field of computable reals is more intriguing.

Computable numbers are real numbers that can be computed to any desired precision using a finite, terminating algorithm. Emile Borel introduced the concept of a computable real number in 1912, based on the intuitive notion of computability available at the time.

A real number a is considered computable if it can be approximated by a computable function

$$f : \mathbb{N} \to \mathbb{Z}$$

in the following way: for any given positive integer n, the function produces an integer $f(n)$ such that:

$$\frac{f(n)-1}{n} \leq a \leq \frac{f(n)+1}{n}.$$

The fact that computable real numbers form a field was first proved by Henry Gordon Rice in 1954 (Rice 1954).

Therefore, playing a constant in this model would mean selecting the index of the computable function. However, despite this scenario being theoretically feasible, an actual game is impossible.

5.2.4.2. On the theory used in the game

In the online game, only limited decimals are used, meaning those with finitely many digits after the decimal point in base 10. More specifically, decimals with a fixed maximum length are used. The relations in our language are >, <, and =. As for functions,

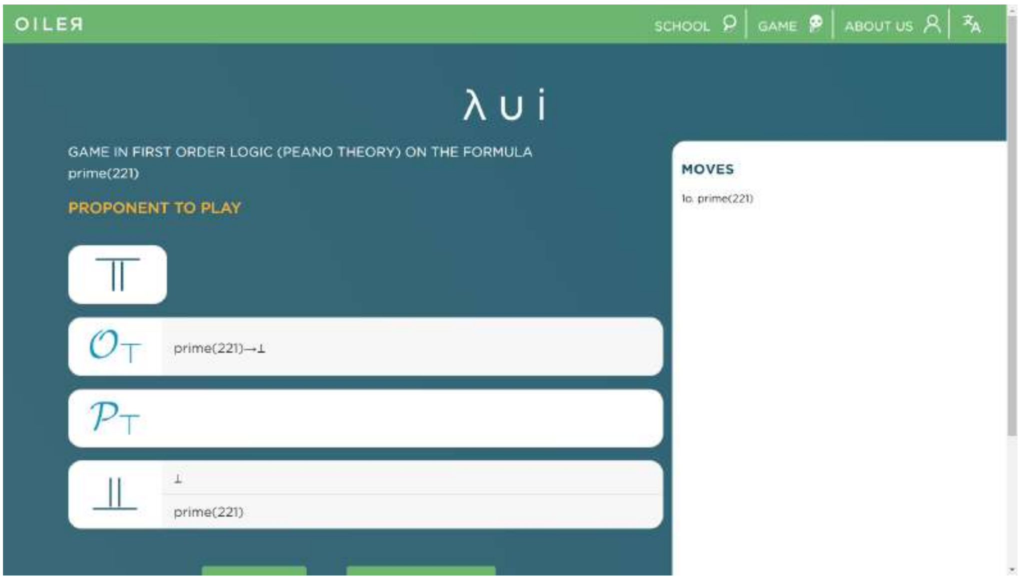


Figure 5.15.: The game is played on the formula PRIME(221).

we consider all the commonly used functions in analysis: $+$, $-$ (binary), $-$ (unary), $/$, $\times$, pow, log, sin, cos, tan, $\sqrt{x}$, $\sqrt[3]{x}$, absolute value, π, e, and ϕ. Clearly, there are formulas that are true in $\mathbb{R}$ but false when restricted to our model. Furthermore, the relations of $>$, $<$, and $=$ turn out to be decidable in our model, whereas they are only semidecidable in computable reals[8]. However, this does not impose a pedagogical limitation on our game, because the formulas from which players can choose are almost always formulas with the same truth value in both models. And when this is not the case, interesting educational insights can be drawn from the discrepancy. Furthermore, we believe that the set of limited decimal numbers is sufficient at an educational level to provide the students with the necessary intuitions for understanding real numbers. The model of limited decimal numbers captures the underlying dynamics and conveys that every real number can be approximated with reasonable accuracy by a limited decimal, an accuracy that clearly increases with the number of significant digits available.

We emphasize that the functions on closed terms are automatically calculated by the computer[9], and so is the truth value of closed relations: in other words, the SHORTCUT mode is always active for every predicate. Consequently, since there are no other predicates defined from the basic predicates, **T** turns out to be empty, and therefore not present during the game.

To conclude this section, we highlight a fact of crucial importance: unlike what

[8] In computable reals, if two numbers are different, the computation will eventually identify this difference. However, in general, it cannot determine if two reals are equal. Conversely, if $>$ and $<$ are satisfied, the computation will eventually realize it, but if they are not, it might never become aware of this.

[9] Reconstructing the logical steps necessary to justify, for example, the sum of real numbers every time would be excessively demanding and not in line with the goals of the game.

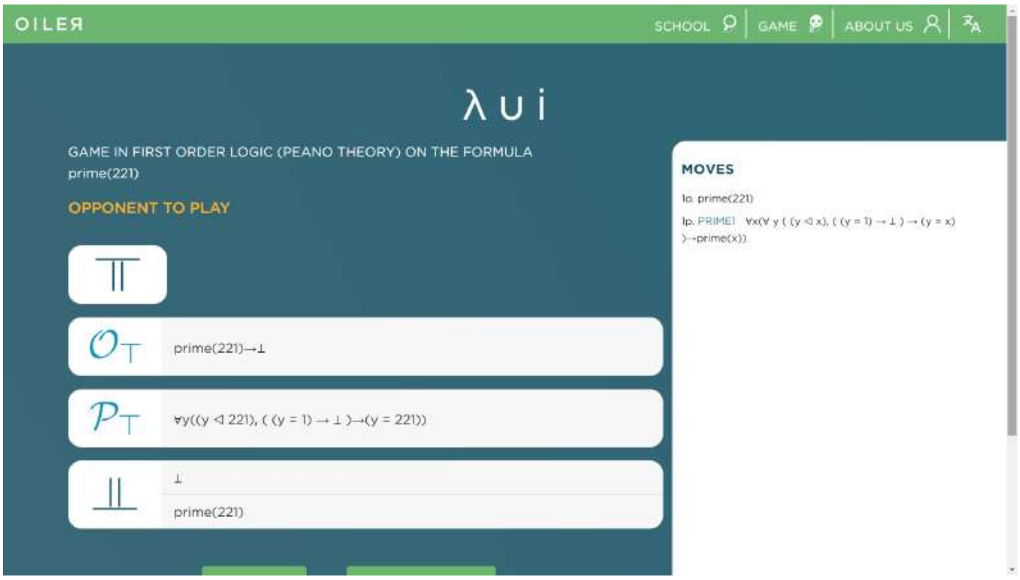


Figure 5.16.: **O** must provide a number y that satisfies the formula $\forall y(y \lhd 221,\ (y = 1 \to \bot) \to y = 221)$.

happens in PA, the domain of many functions is not the entire set of $\mathbb{R}$. The most straightforward solution is that, as soon as the computer gives a domain error, the last player who made a move is asked to change the numbers they have chosen. Unfortunately, this is not a valid solution: consider, for example, $log(x \times y)$ and suppose that at some point in the game a player chose $x = 0$. The player subsequently called to choose y cannot select any value due to the previous choice of x. In other words, although not responsible for the domain error, the second player bears its consequences.

The solution we have found is twofold: on one hand, in every playable formula, the domain is always precisely specified (by including the domain conditions as hypotheses in the formula). On the other hand, when the computer returns a domain error, the function is simply not computed, but the game continues with the unprocessed expression: the player responsible for the domain error will lose because the number they chose does not meet the pre-established domain conditions.

5.2.4.3. The Game

In the AUOQUAMEL game mode, users choose which formula to play from 21 available options. The rules of the game are identical to those of the others modalities, with the only exception being that the closed terms players can play are indeed limited decimals. The chosen formulas allow for a gradual approach to the concept of limits, with the number of quantifiers and connectives progressively increasing. The first six formulas pertain to the concept of bounded and unbounded functions. Clearly, some are true while others are not. It's interesting to note that a function f is upper-bounded if $\exists x \forall y(f(y) < x)$, while it is unbounded above if it satisfies $\forall x \exists y(f(y) > x)$, which

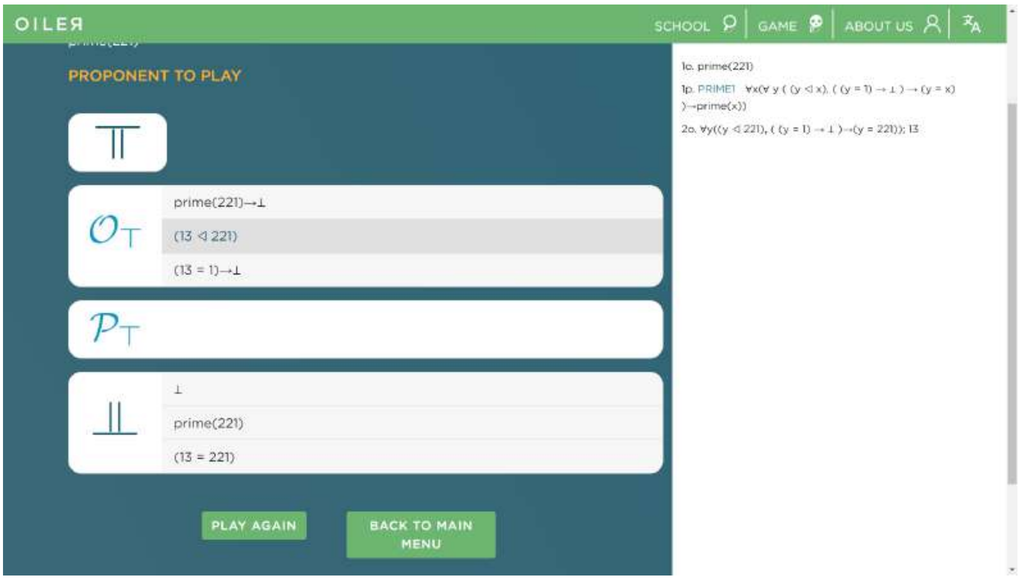


Figure 5.17.: **O** claims that 13 divides 221 even though $13 = 1$ and $13 = 221$ are both false.

is the negation of the previous definition; logically, the difference manifests in the swapping of quantifiers. The next six formulas, on the other hand, refer to functions that are bounded and unbounded within an interval. Logically, this involves adding an implication compared to the previous formulas, in the hypotheses of which the interval is specified. The last 9 formulas concern limits, which require, on a logical level, the addition of a third quantifier.

Let's now show an example of a play on formula number 3, namely $\exists x \forall y(\sin(y) < x)$. In other words, the formula asserts that the sine function is bounded. The formula, written in normal form, is $(\forall x(\forall y(\sin(y) < x) \to \perp)) \to \perp$.

O begins by claiming that the sine function is actually unbounded: $\forall x(\forall y(\sin(y) < x) \to \perp)$ (Figure 5.18).

P claims that the sine function is indeed bounded, stating that **O** will not be able to find a y for which $\sin(y)$ will be greater than 3 (Figure 5.19).

O is thus called upon to find a value y for which $\sin(y) \geq 3$. As shown in Figure 5.20, **O** chooses the number 0.

P wins by pointing out that $0 < 3$ is, in fact, true (see Figure 5.21 and Figure 5.22). Let's remember that the only mode available for AUOQUAMEL is indeed the shortcut mode.

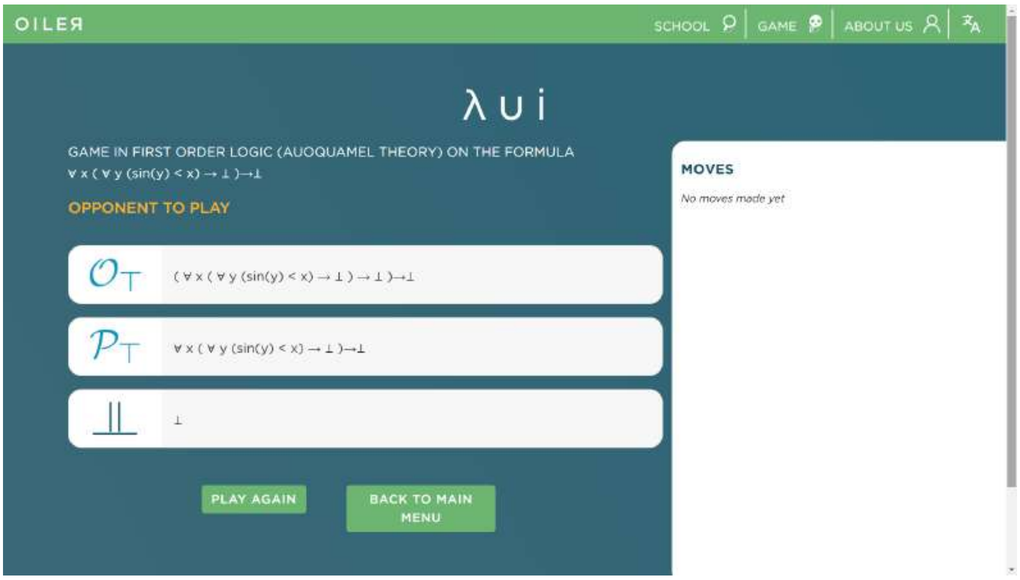


Figure 5.18.: **O** selects the only available formula in $\mathbf{P}_{\top}$.

5.3. Further Developments

5.3.1. Alfred Tarski

The reader may have noticed that we have not dealt with Euclidean geometry, as the software has not yet been implemented in this direction. We limit ourselves here to providing an intuition on how the game will be structured. Tarski's theory is a first-order formal theory for Euclidean geometry. The formalization, rather elegant, involves variables referring only to points, no function symbols (in particular, no constants) in the language, and the use of only two predicates besides equality: the ternary predicate *betweenness* $\beta(x, y, z)$, indicating that point y is aligned and lies between x and z, and the quaternary predicate *distance* $\delta(x, y, z, w)$, indicating that the distance between points x and y is equal to the distance between points z and w.

As can be immediately understood, the absence of closed terms makes the theory unsuitable for an immediate transposition into the game. However, the work of M. Beeson (2015) proves extremely useful, providing tools to make Tarski's theory constructive. An idea, evolving from Beeson's work, is that players can introduce constants to play with (*i.e.*, ordered pairs of decimal numbers) in a Cartesian environment, in the style of dynamic geometry software, with typical constructions that in these softwares are allowed.

5.3.2. AI and λuì

In our discussion, we have outlined several adjustments to enhance the game's suitability for human interaction. However, a computer engaged in strategy research is not subject to the boredom that we aimed to reduce with these modifications. Referring,

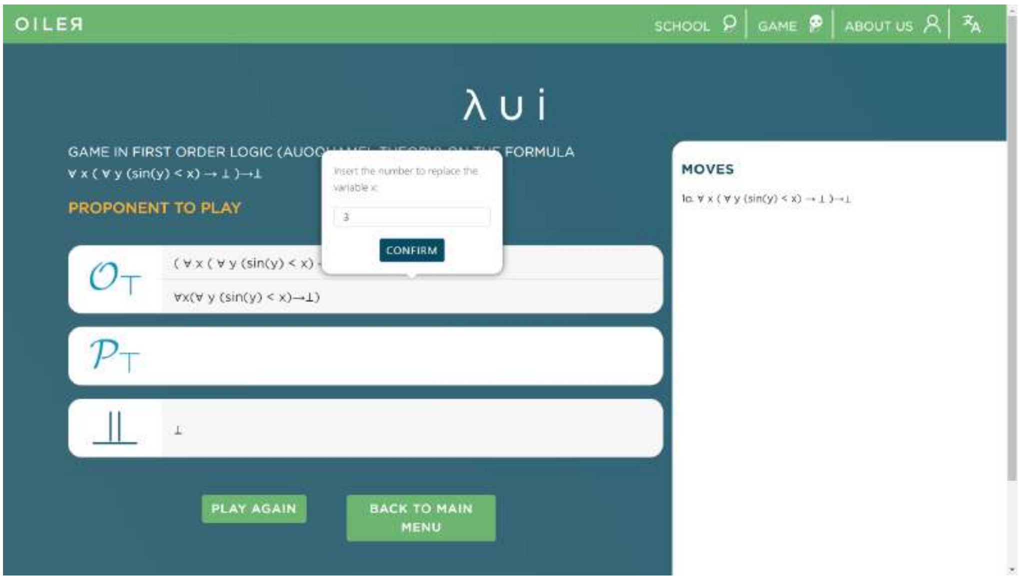


Figure 5.19.: **P** chooses the constant 3 to replace the variable x.

for example, to PA, the game can be implemented by requiring **O** to make exclusively generic moves, as described in 3, and by inserting all the axioms into $\mathbb{T}$, including the schema of the induction principle. At that point, a winning strategy for **P** would correspond to an actual proof, and we could view λuì as a proof-search program. In this direction, the possibility for users to create their own theories and formulas will be added.

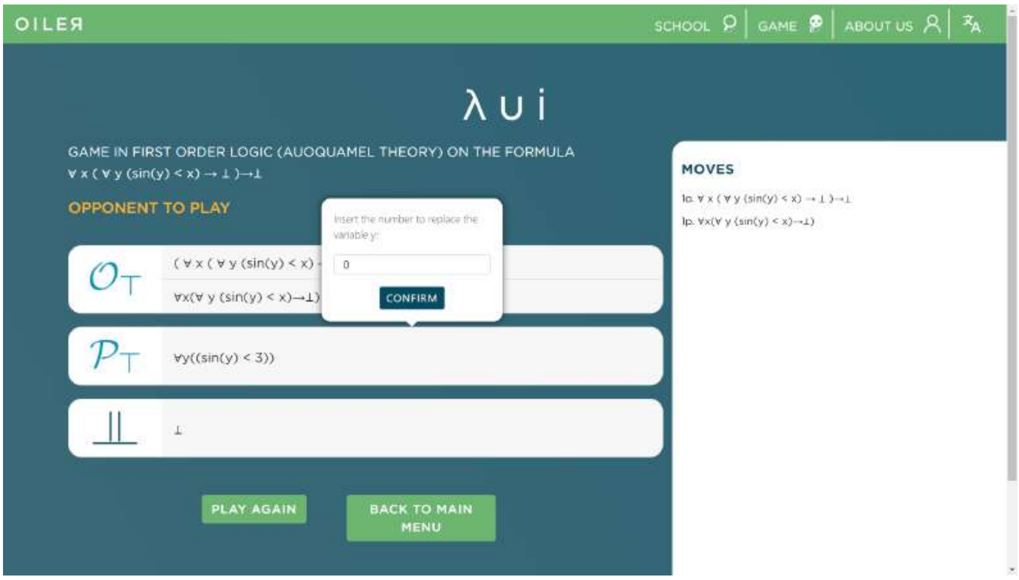


Figure 5.20.: **O** chooses the constant 0 to replace the variable y.

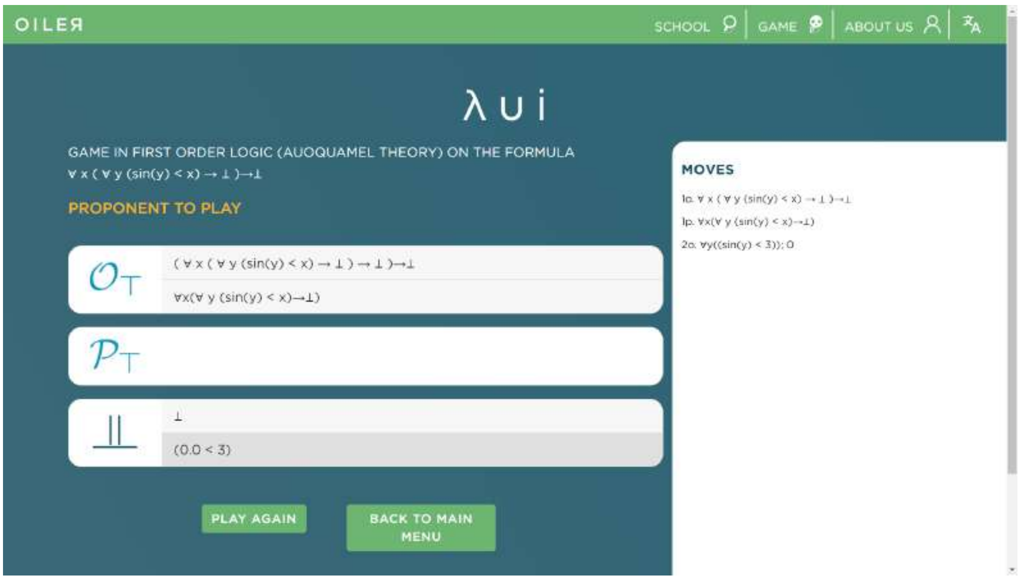


Figure 5.21.: **P** clicks on $0 < 3 \in \perp\!\!\!\perp$.

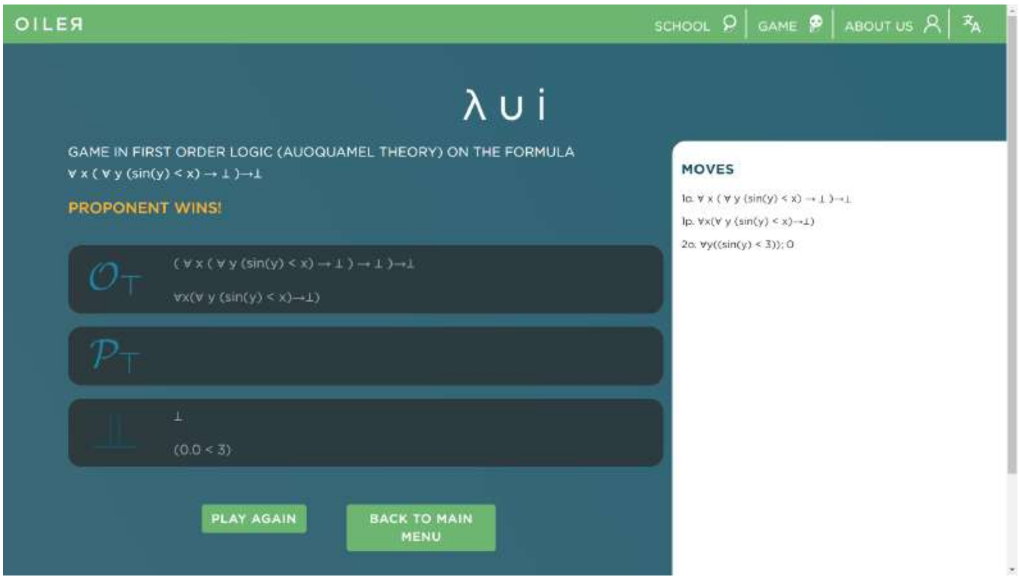


Figure 5.22.: **P** wins.

6. Logic Education

6.1. Preliminary Definitions

In the previous chapters, we presented a precise correspondence: derivations in LK and winning strategies in $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ are just different representations of the same mathematical object. It is clear that different representations can be of use in different contexts, as—even though they describe the same object—each highlights certain aspects, thus facilitating the intuition of those who engage with a specific representation. The purpose of this chapter is to determine whether the game-like representation $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ is useful in Mathematics Education (ME). Before proceeding, at the expense of logical elegance, we need to clarify and simplify the dynamics of the $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ game to make it suitable for an educational analysis.

Let us start with the language. The minimal language $\mathscr{L} = \{\forall, \to, \bot\}$ brings about considerable simplifications at the logical level and—in the previous chapter—an epistemological justification was even attempted for Krivine's normal form. However, typical mathematical activity involves the explicit use of all quantifiers and connectives and accepts any formula written correctly at the syntactic level, without requiring specific forms for the formulas. The language we will consider from now on is therefore $\mathbb{L} = \{\forall, \exists, \wedge, \vee, \to, \neg, \top, \bot\}$ where $\bot$ and $\top$ are respectively two symbols for predicates to indicate false and true. Their interpretation is constant in every model.

The game $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ is played between two players, the *Proponent* and the *Opponent*, who hold opposite views on a certain formula F. In particular, the Proponent thinks that F is true while the Opponent thinks that it is false; in other words, the Opponent asserts $\neg F$ or, equivalently, $F \to \bot$.

Clearly, one can formally play the game $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ using the extended language just presented: it will suffice—before playing—to translate the formula into the minimal language and then into normal form, which is always possible as shown in Chapter 2. By doing so, we can extend the rules of the game $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ to every quantifier and connective of our new language.

Definition 30 (Dialogue Rules)**.** *Let F be a statement written using connectives and quantifiers of $\mathbb{L}$. Then, a dialogue between* **P** *and* **O** *on the formula F is conducted according to the following rules.*

- *If a player asserts $\top$, they win; if they assert $\bot$, they lose.*
- *If a player asserts $\neg F$, then the other must assert F.*

- *If a player asserts $\forall x F(x)$, then it is up to the other player to find a counterexample, that is, a closed term t for which $\neg F(t)$.*
- *If a player asserts $\exists x F(x)$, then it is up to them to provide an example, that is, a closed term t for which $F(t)$.*
- *If a player asserts $F \wedge G$, then it is up to the other player to state which among F and G is false.*
- *If a player asserts $F \vee G$, then it is up to the asserting player to state which among F and G is true.*
- *If a player asserts $F \to G$, then the other must state that F is true while G is false.*

These rules correspond to the usual rules of Game Semantics (Thomas Barrier 2008). We recall that these rules are not contrived; they are those implicitly present in any mathematical proof. Indeed, they are closely related to the rules discussed in Chapter 2. Moreover, although these rules may appear to the reader as intuitionistic, when properly framed—as in our game $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$—these rules can also work well for classical logic.

Let us spend a few more words on dialogue rules. We note that the purpose of every dialogue is to progressively decompose the statement into ever simpler statements until we reach something whose truth value is shared. A key example to understand this dynamic is when one asserts $A \wedge B$. Clearly, to assert a conjunction means to be able to assert A or to assert B, depending on the Opponent's will. From this dialogue rule, we can deduce the rules on $\vee$ and on $\to$. It is also important to note that these rules are typical of scientific and rational reasoning in general, not just mathematical.

The game can be played within a theory that states the set of formulas, referred to as **T**, which both players agree to be true, *i.e.*, the axioms. The theory also specifies the functions, and consequently the closed terms, that can replace the variables, and the basic relations, thereby specifying the atomic formulas from which a formula is constructed.

So, what is a formula? A formula is a mathematical statement written in a mathematical language that can possibly be proven within a theory; in such cases, it is referred to as a *theorem* of that theory. Every statement presented at any educational level can be translated into formal language, and there is no theorem presented at any grade or university course that cannot ultimately be traced back to a formula. We are here referring to a wider concept of a theorem that also includes algebraic formulae, Euclidean geometry theorems, analytical geometry and analysis statements. During regular mathematical activities, including in an educational context, what is typically referred to as a theorem is just a provable formula that is deemed important. Indeed, in a logical sense, a theorem is not only a lemma or corollary but any provable statement, such as "3 is odd". Similarly, some analysis theorems are merely granted the status of a property, such as "the derivative of a sum is the sum of the derivatives".

In the next section of this chapter, we aim to establish a comparative dialogue between the concepts of *proof* in logic and *proof* in ME. We intend to examine some definitions of *proof*, *proving*, and *argumentation* in ME, and attempt to compare them with what occurs in logic. By means of isomorphism, when discussing anything related to proof in logic, we will refer to the game $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$, specifically to the simplified version just introduced. Additionally, we will define the following to facilitate our analysis:

- *Dialogue Rules*: the rules of the game $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$, extended to every quantifier and connective (Definition 30).
- *Argument*: a partial strategy in the game. If the partial strategy benefits the player using it[1], we will use the term *valid argument*. A valid argument is known in Game Theory as a tactic.
- *Argumentation*: a play where at least one of the two players uses an argument at least once. In other words, a non-entirely-random play. Clearly, carrying out an argumentation can lead to the formulation of new arguments.
- *Derivation*: a winning strategy in the game for the Proponent.
- *Derivation Rules*: logical rules for communicating a winning strategy, such as the rules of $\mathrm{LK}_{\mathrm{game}}$ or the rules of LK .
- *Proof*: a winning strategy effectively communicated both from a mathematical standpoint—through *Derivation Rules*—and a social standpoint, that is, from the perspective of dialogue with the community by which this proof must be accepted. This definition aligns with the definition of *proof* given by Durand-Guerrier, Boero, Douek, et al. (2012), defined as a cultural product subject to constraints of consistency [derivation rules] and communication [social rules].
- *Theorem of* $\boldsymbol{T}$: a formula F for which there is a proof in the theory $\mathbf{T}$.
- *Conjecture on* F: hypothesizing that the Proponent has a winning strategy on F and, through argumentations, trying to build arguments in favor of one's hypothesis.

We conclude this brief introductory paragraph by emphasizing that viewing proof as a winning strategy in a dialogue is not unnatural. Szabóo (1967) argues that the deductive logic of Euclid originates from pre-Socratic dialectics, with conversation serving as the model. Indeed, according to P. Ernest (1991), the mathematical proof dates back to classical Greece, reflecting the emerging dialogical nature of public life and of the state. We can indeed trace back "the source of deductive mathematics and logic to dialectical argument and disputation".

[1] As far as the Proponent is concerned, a strategy is said to "benefit" the Proponent if it leads to victory in specific instances or if it forces the Opponent to make certain concessions, namely adding formulas to the set $\mathscr{A}$.

6.2. Proving in Mathematical Logic *vs* Proving in Mathematics Education

In ME, argumentation, proving and proof “are concepts with ill-defined boundaries”(Hanna 2014). Nevertheless, or maybe precisely because of this, there is a vast amount of literature dedicated to the comparison between *proof* and *argumentation*, with frequent clarifications and redefinition of these terms. We will attempt a brief discussion here, relying—in the first part of this section—on the review presented by A. Mariotti (2006).

Harel and Sowder (1998) define the “process of *proving* (of which a proof can be considered a product)” as “the process employed by an individual to remove or create doubts about the truth of an assertion. The process of proving includes two subprocesses: *ascertaining* and *persuading*”. Ascertaining is the process an individual employs to remove their own doubts, whilst persuading is the process employed to remove others’ doubts.

The distinction between *ascertaining* and *persuading* holds significant relevance in social, educational, and pedagogical realms. This is because the rules and strategies one employs in self-reasoning might differ from those used in a discourse with others. On the one hand, this distinction is less pronounced in a logical context, where Harel and Sowder themselves argue that “arguments must simultaneously serve both ascertaining and persuading purposes”. On the other hand, the very essence of their definition, rooted in the duality of *removing or creating doubts*, aligns closely with the argumentation process we defined: the Proponent aims to remove doubts about F, while the Opponent tries to create doubts about it.

A. Mariotti (2006) then discusses the perspective of Fischbein (1982), who argues that—perceptually speaking—after a proof has been presented, further checks seem desirable to confirm its validity. As an aside, it is worth noting here that Fischbein appears to view these "checks" as tools useful for verifying a proof, not for constructing it. Fischbein identifies an ontological discrepancy in this phenomenon between empirical verification (common in everyday behavior) and deductive reasoning (characteristic of theoretical behavior), recognizing it as a source of difficulty for students. Expanding on this, Mariotti mentions that this discrepancy has been further explored and accentuated by Duval (1989) and Duval (1992), who sees a clear opposition between argumentation and proof.

According to Duval, the split between two levels—the semantic and the theoretical—might be so profound as to be irreparable. In this view, the notion of proof, when seen as a process aiming to persuade another person, could be at odds with the standards of a mathematical proof. This distinction is particularly valuable in an educational context. Without precisely establishing the Dialogue Rules, there is a risk that during argumentation, one might resort to “rhetoric means”(Duval 1989), which may stray far from mathematical formalism. What Duval’s words seem to implicitly suggest, and what the game $\mathbf{T}\mathscr{UVA}$ explicitly endorses, is the importance of focusing on the Dialogue Rules *before* delving into Derivation Rules. Jumping straight to Derivation Rules without first addressing Dialogue Rules can be equivalent to, for instance, urging students to find strategies for the game of tic-tac-toe without first explaining its rules or letting them play a few games. Once the Dialogue Rules are set, the gap between the semantic and the theoretical becomes less pronounced, as the theoretical Deduction Rules (upon which a strategy's exposition is based) come after—and are built upon—the Dialogue Rules.

In this context, various studies have tried to define the status of argumentation in mathematics, looking for potential continuity between argumentation and proof rather than a rupture between them. "The hypothesis of cognitive unity suggests that in certain cases this argumentation [developed to produce a conjecture] can be used by the student in the construction of a proof by organizing a logical chain of some of the previously formed arguments"(Pedemonte 2007; Boero, Garuti, E. Lemut, et al. 1996). Consequently, numerous studies have centred on the formulation of conjectures (Boero 1994; Boero, Dapueto, Ferrari, et al. 1995) and the associated argumentations. Often, the phase of producing a conjecture showed "a rich production of arguments that aimed to support or reject a specific statement", and it was possible "to recognize an essential continuity between these arguments and the final proof" (Mariotti 2006).

It is now interesting to revisit this discussion in light of the definitions provided in the Section 6.1. Indeed, in formulating a conjecture—that is, guessing if the Proponent might have a winning strategy—argumentations are essential, *i.e.*, games in which different arguments are put to the test. Moving on to the "subsequent statement proving stage, the student [...] organizes some of the arguments [partial strategies, what we referred to as *valid argument*] into a logical chain"(Boero, Garuti, E. Lemut, et al. 1996, p.113), thus deriving a total and winning strategy, which is a derivation.

Indeed, looking for a winning strategy in a game is something natural, but the need only arises as a consequence of the interest one feels in the game, which in turn may develop only through playing. Presenting a proof for F without first engaging in a dialogue about F is as unappealing as solving a chess puzzle would be to someone who has never played chess.

When conjecturing, both in our meaning and in ME, a particular statement F is put to the test, with the role of the Proponent being played in the game on F. In an educational context, the Opponent could be the teacher, another student or group of students, or even the student themselves. In this regard, the analogy proposed by Balacheff (1999) fits well: "argumentation is to a conjecture what mathematical proof is to a theorem". Indeed, the act of argumentation (playing against another and testing one's arguments) is a process of conjecture, whereas executing a winning strategy on a formula F is a process of derivation, and—in a way—of proof, especially if the Opponent accepts the truths established by the game $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ as mathematical truths. Nevertheless, although continuity in content is often recognizable, it sometimes happens that the construction of a deductive chain that correctly relates the theoretical elements involved may be difficult to achieve (Mariotti 2006). In fact, having a Derivation does not necessarily mean being able to present that derivation with Derivation Rules.

A. Mariotti (2006) states that research studies consistently highlight *the need for* *an early start in proving practice.* In line with this sentiment, important work can be observed even at the primary school level. A feature evident across most of these projects is their shared emphasis: the attempts to foster the development of mathematical meaning are widely based on thoughtful mathematical activities investigated by children. In this case one of the basic aims is very often that of establishing a "mathematical community in the classroom" (Bartolini Bussi 1998; Arsac 1992; Maher and Martino 1996; Yackel and Cobb 1996). This concept of a "mathematical community" echoes the idea of a group that aligns with Dialogue Rules before delving into Derivation Rules.

It is intriguing how Mariotti (2006), at the chapter's conclusion, speaks of the "metaphor of the game" which, for us, has evolved beyond being a metaphor: *it is a full abstraction*! Mariotti observes that when students engage in game, they develop strategies. This implies that actions are cherry-picked from a range of possibilities based on intuitive or logical reasons. Feedback from the environment enables the learner to assess the effectiveness of their chosen strategy, guiding them towards its acceptance or rejection. The sequence of interactions between the student and the environment (or the 'Opponent', in our terms) forms what is called the "dialectic of action". As the game progresses, the student navigates the "dialectic of formulation", a process focused on "gradually establishing a shared language" that facilitates the elucidation of actions and methodologies. During this phase, according to Brousseau's model (Brousseau 1997), one student's propositions may be discussed by another student—not from the point of view of the language, but from the point of view of the content (that is to say, its truth). Such debates regarding the effectiveness of strategies are commonly called "validation phases". The dialectics of contradiction and the emphasis on counter-examples can be understood in this light, as introduced and subsequently elaborated upon by Balacheff (1985), Balacheff (1987), and Balacheff (1991). Both of the insights provided by Brosseau and Balacheff fit seamlessly into our more formal framework.

Let us now shift our attention to the overview presented by Stylianides, Bieda, and Morselli (2016). Firstly, the "proof in the context of a classroom community" is defined as "a mathematical argument for the truth or falsity of a mathematical statement that meets both of the following criteria":

- An argument qualifying as a proof should use true statements, valid modes of reasoning, and appropriate modes of representation, where the terms "true", "valid", and "appropriate" are meant to be understood with reference to what is typically agreed upon nowadays in the field of mathematics, in the context of specific mathematical theories.

- An argument qualifying as a proof should use statements, modes of reasoning, and modes of representation that are accepted by, known to, or within the conceptual reach of students in a given classroom community.

This definition appears to fit neatly within our theoretical framework, where the first criterion mirrors all that is related to derivation, while the second criterion refers to the community and the communication of the proof. Moreover, there is an explicit emphasis on the need for the community to share the "modes of reasoning" (*i.e.*, Dialogue Rules) used in a derivation.

According to the overview in (Stylianides, Bieda, and Morselli 2016), many researchers rely on Toulmin (1958) to analyze discussions among their students. This model did not originate within the mathematical or logical domains but is, in fact, a legal model. With Toulmin, there can be no bridge between argumentation and proof, as the environment he employed is completely detached from a logical setting[2].

To conclude, we add that approaches to proof based on Game Semantics are not uncommon, with an interesting example found in (Arzarello and Soldano 2019). However, in this specific case, Hintikka's original methodology is used, which is less formal and structured compared to Krivine's approach. As one can observe, formality plays a crucial role in what we provide.

6.3. Can students tell truth from falsehood?

A shared characteristic of the definitions provided above, whether in a logical or ME context, is the importance attributed to the concepts of conjecture and argumentation. As a direct consequence, it is also evident that utmost importance is given to the notions of truth and falsehood, and their inherent duality: a prerequisite for having a proof-proficient student is to have a student capable of handling both true and false statements.

The significance of dialogical learning, based in the contrast between truth and falsehood, is emphasized by various studies (McLaren, Adams, and Mayer 2015). Recent research has indeed shown that presenting exercises completed incorrectly alongside exercises completed correctly (Rushton 2018) has led students to an increased understanding of mathematics. Additionally, as highlighted in (D'Amore, Fandiño Pinilla, Marazzani, et al. 2023), there is an abundant international bibliography on impossible problems, with (Schubauer-Leoni and Ntamakiliro 1994) cited as a foundational example.

However, at the school level, the balance consistently tilts in favor of true statements. One can even be led to question whether students truly possess the ability to handle both true and false statements, recognizing them as equally informative. Our hypothesis is that this is not always the case and an explicit focus on truth and falsehood is essential from the early years of primary school, as the competency to manage truth and falsehood does not develop on its own.

6.3.1. Quantitative Evidence: a Large-Scale Test

We present the results of an INVALSI item (see Figure 6.1) for Grade 5, corresponding to the last year of primary school, which was proposed to approximately 15 000 students

[2]This model does not seem to provide the ME community with a solid foundation to analyze proofs and argumentations in mathematics.

in the academic year 2022/2023.

Two characters, Avac and Afru, provide information about a polygon. Avac is a character who always tells the truth, while Afru is a character who always lies. Even though we will delve deeper into some aspects related to the famous characters of the knight and the knave in (Smullyan 1987), it is worth noting that Avac, when read backward, becomes *cava* (in Italian, "cavaliere" means "knight"), whereas Afru, when read backward, becomes *(f)urfa* (in Italian, "furfante" means "knave"). Both characters describe the same polygon from those presented, providing details that allow its identification.

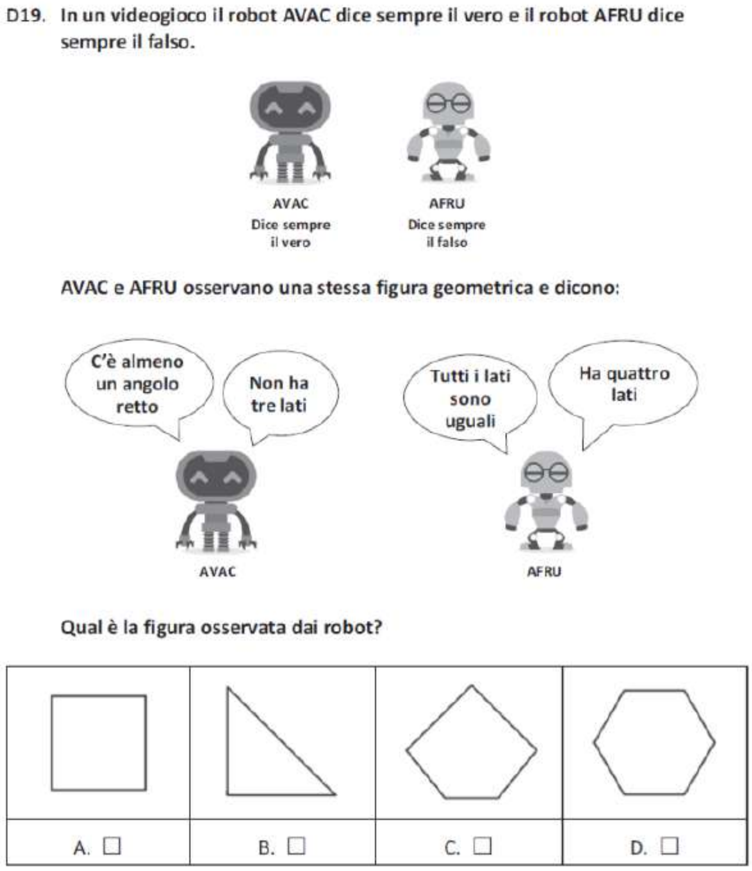


Figure 6.1.: Item D19 of INVALSI Grade 5, 2023

Avac indicates that the polygon has at least one right angle and does not have three sides. These statements rule out both the regular hexagon, which lacks right angles, and the right-angled triangle, which has three sides. On the other hand, Afru claims that the polygon has all congruent sides and a total of four sides; each of these statements, being false, allows us to discard the square and identify, by exclusion, the pentagon with three right angles as the polygon that meets the given descriptions. The correct answer is therefore C.

As seen from in Table 6.1, 45.7% of the students got the correct answer, while the distractor A was the most effective, with a response rate of 39.8%. The distractor A is constructed by considering as true all four statements made by Avac and Afru: the polygon has at least one right angle, does not have three sides, has all sides equal, and has four sides.

Missing	A	B	C	D
2.4%	39.8%	5.3%	45.7%	6.9%

Table 6.1.: Item administration results.

Table 6.2 provides additional information about the item and its significance within the test. In an INVALSI test, students are divided into three groups based on their performance in the test itself: high performance, medium performance, and low performance. An important factor to consider when evaluating an item is the Total Cor, which indicates the discriminative power of the item. It answers the question "Is it true that high-performance students generally answered better on this question than low-performance students?". More specifically, the Total Cor of a question is calculated by normalizing between -1 and 1 the difference between the number of high-performance students who answered the question correctly and the number of low-performance students who answered the question correctly. A negative number in Total Cor indicates that the question yielded a result counter to the trend of the rest of the test, while a Total Cor close to 0 indicates that the item is not discriminative. In the case of Avac and Afru, the Total Cor indicates that the item is indeed discriminative.

The labels 1, 2, 3, 4 respectively indicate the answers A, B, C, D. The label 7 indicates an unclassifiable response, and 9 indicates a missing answer. The Pt Bis is another interesting factor, which indicates where the students that answered option x stands in relation to the average performance: a negative factor is below average, while a positive factor is above average.

item:22 (D19)

N 16828 (Weighted 16828) Item-Rest Cor. 0.41 Item-Total Cor. 0.46
Obs mean: 0.46 Exp Mean: 0.46 Adj Mean: 0.46 (Exp & Adj use: PV)
Item Threshold(s): -0.05 Weighted MNSQ 1.00
Item Delta(s): -0.05

Label	Score	Count	% of tot	Pt Bis	t	sig	PVAvg:1	PV SD:1
1	0	6693	39,77	-0,28	-38,5	0,000	-0,665	0,921
2	0	889	5,28	-0,11	-14,7	0,000	-0,787	0,970
3	1	7695	45,73	0,41	59,1	0	0,261	1,031
4	0	1161	6,9	-0,11	-14	0	-0,728	0,992
7	0	194	1,15	-0,08	-9,8	0	-1,005	0,901
9	0	196	1,16	-0,06	-7,9	0	-0,875	1,043

Figure 6.2.: The table presents some statistics on the question, including its discriminatory power.

6.3.1.1. A Qualitative Analysis of the Item

First and foremost, let us delve deeper into the question, attempting to discern, where possible, the component related to geometry from the purely logical one. Two challenges seem to emerge at the geometric level, one certainly more prominent than the other. Firstly, the correct polygon (answer C) is not a conventional shape in didactic practice: it is a pentagon that is neither equilateral nor equiangular. Moreover, the three right angles of the pentagon are not immediately identifiable because, once again, there is sometimes a tendency in the educational environment to primarily show angles with a horizontal side.

Now, let us put ourselves in the shoes of student X with good logical abilities but average geometric skills, which are insufficient to effectively address the two challenges mentioned above, and attempt to imagine their performance.

Avac states that the figure has a right angle and does not have three sides: at first glance, the shape that appears to match these properties is the square. If lazy, our student might stop here and simply select answer A. Perhaps this dynamic describes some of the 39.8% cases, but likely not many; there is another, more complex scenario we will consider later where a student only reads Avac's statements. Our student X continues reading and finds out that Afru says "all sides are equal" and that "it has four sides". At this point, our student must deviate from their prior assumption: Afru's statements suggest that the answer is not the square nor the regular hexagon. Without perhaps fully understanding the reason behind it, our student will have no choice but to select answer C.

By contrast, consider student Y, with excellent geometric skills but weak logical abilities. The student, after reading Avac's statements, concludes that the figure in question must necessarily be the square or the pentagon. However, they fail to interpret Afru's statements as false, concluding that the right polygon is the square.

This contrived comparison between the types of students was an attempt to argue in favor of a hypothesis: within that 39.8%, there are probably many students who do not know how to handle false statements or who, perhaps even worse, believe these statements are useless. Earlier, we said that, aside from laziness, another reason could lead one to not read Afru's statements: *the belief that these play no role and are therefore non-informative because they are false*. Indeed, a student has every right to hold such a belief, as false statements are almost never adequately addressed in the educational environment.

A last comment on the item before proceeding further, regarding Afru's statements. To answer the question correctly, one does not need to be able to interpret both statements correctly: either of the two—combined with Avac's statements—leads to the correct conclusion. Let us analyze the statements in more detail. The statement "All sides are equal" is very interesting because it is as easy as it is difficult to negate. It is easy because many students, if faced with any polygon, would be able to say—perhaps after some measurements—whether the property "All sides are equal" is satisfied by the given polygon. On the other hand, precisely enunciating the negation of this statement is a very complex task (we would bet that there would be few correct answers both in secondary school and at the university level), and the problem mainly lies in the property "equal". Unfortunately, common language does not do justice to a fundamental distinction, and two sentences like "All sides are blue" and "All sides are equal" have the exact same structure.

Nevertheless, while "being blue" is a unary predicate, which predicates the property of an object being blue, "being equal" is a binary predicate, which predicates the property of a pair of objects being equal (equal in a given context). Logical language, unlike natural language, highlights this distinction very well: "All sides are blue" is expressed as $\forall$side (BLUE(side)) while "All sides are equal" is expressed as $\forall$side1, side2 (EQUAL(side1, side2)). The correct negation is therefore "There is at least one pair of different sides".

The second statement by Afru, on the other hand, is straightforward to negate: the figure in question does not have four sides. While there can be a debate on the difficulty of accurately negating Afru's first statement, proponents of the ideology "there is no need for explicit logical activities because logical skills are developed autonomously through other teachings" will struggle to explain why 39.8% of fifth-grade students could not negate the phrase "it has four sides" and interpret the negation correctly.

The two students X and Y described are not atypical students. They simply lean towards one or the other of two typical aspects of geometry: the figural and the conceptual (Mariotti 2005). This dual aspect of geometric knowledge led (Fischbein 1993) to formulate the theory of *figural concepts*: mental objects possessing both conceptual and figural properties simultaneously. Teaching must gradually lead the student to structure sensory experiences, resulting in a fusion of concept and figure:

“A last remark refers to the possibility to practice, with the students, mental activities in which the cooperation between the figural and the conceptual requires a special endeavor. In such activities, the student has to learn to mentally manipulate geometrical objects by resorting simultaneously to operations with figures and to logical conditions and operations.” (Fischbein 1993, p. 158)

Fischbein hopes for situations in which “the relationship between logic and figural aspects is explicitly applied”(Fischbein 1993, p. 156). Consequently—Fischbein continues—one aim of geometry education should be to create types of situations where a close cooperation between the two aspects is systematically required, ensuring that the figural component of the concepts does not elude logical control and leads to misconceptions and errors. The analysis we have carried out seems to suggest that the resolution error in the question is due to a similar dynamic, where the logical component—as discussed in the previous section—is the most relevant one.

The question we considered was answered incorrectly by about 50% of fifth-grade primary school students and—despite its artificial characters that always tell the truth or always lie—it is not contrived. Beyond the specific geometric environment, the question involves competencies about which it would be desirable to have broad consensus: *the ability to handle simple statements, both true and false, that possibly contain quantifiers.*

6.3.2. Knights and Knaves as an Educational Tool

What obstacles can hinder a satisfactory treatment of falsehood? In this section, we address a critical issue in dealing with falsehood in a classroom, and suggest how the characters of the knave and the knight can assist in restoring a balance between truth and falsehood.

The island of knights and knaves is a well known tale by Raymond Smullyan, describing an island whose inhabitants are either knights, who always tell the truth, or knaves, who always lie. This island is mostly known from logical puzzles in which one is tasked with identifying the inhabitants of the island on the basis of their statements, with such puzzles making an appearance as early as primary education (Carotenuto, Coppola, and Tortora 2017). Nonetheless, a story built around characters who speak the truth and characters who lie is likely to have a wider educational value, not just because "putting these matters in human terms has an enormous psychological appeal" (Smullyan 1987), but also because—owing to the use and acceptance of falsehoods (those uttered by the knaves)—it offers a more playful approach to errors, which become part of the learning process rather than being immediately corrected.

To be more precise, we want to highlight a distinction that is usually unspoken in the educational context: the difference between "error" and "falsehood". The word "error" can be interpreted in various ways: for instance, what constitutes an error in everyday life, what is considered an error in mathematics and logic, and the types of emotional reactions an error can provoke. On the other hand, "falsehood" refers to a strictly logical framework, where language and axioms must be clearly defined. The concepts of falsehood and error certainly overlap significantly within a classroom setting: in an educational model where truth is favored over falsehood, and the procedural aspect of mathematics is prioritized over the conceptual, a false statement is often perceived as an error.

Let us expand slightly on what characteristics an error in mathematics can have. Errors can be divided into three broad categories: structural errors (*i.e.*, syntactic), where symbols are used incorrectly (*e.g.*, $3+++==$); errors in meaning (*i.e.*, semantic), where symbols are used correctly but the meaning is incorrect (e.g., $3+3=5$); and errors in interpretation (*i.e.*, pragmatic), where both syntactic and semantic aspects are fine, but the person writing or reading it does not interpret it as established by the community. A statement written with incorrect syntax does not have a truth value (it is neither true nor false), whereas a syntactically correct statement can be true or false. The introduction of the knave breaks the links between falsehood and mistakes: for the knave, making a true statement is an error. In our programme, the terms 'true' and 'false' are indeed preferable to 'right' and 'wrong' when speaking about semantic errors: 'true' and 'false' have a clear logical meaning, and their use helps to *place the two outcomes on equal footing.* Furthermore, we prefer discussions about truth and falsehood to be conducted through role-play, where truth is embodied by knights and falsehood by knaves. We believe that false statements can not only enhance understanding of symbols and concepts[3] but they can especially help bring to the surface the need for argumentation. This richness is lost in the overlap between falsehood and error, where error is perceived as something to be avoided.

[3] To clarify the meaning of the symbol $<$, knowing that $3<2$ is false is as important as knowing that $10>8$ is true.

6.3.3. From the Dialogue Between True and False to Argumentation: the Role of the Teacher

Assuming, for the sake of argument, that the class can navigate between true and false statements, we can—given the theoretical framework just introduced—consider which practices may lead to the emergence of a need to argue. For an argumentation to occur, as extensively discussed, there is a need for two individuals with opposing views on a particular statement.

Referring to the strategy-proof equivalence that we have discussed so far, imagine how boring a game of Tic-Tac-Toe would be if both players played with the X and aimed to get three Xs in a row. Two-player games are interesting and captivating precisely because the two players have opposite goals: one wants to win by—consequently—making the other lose. Each one wants to achieve their goal before the other achieves theirs.

What is usually observed in a classroom setting, however, is that the teacher corrects the student *only* if they have actually made a mistake. Stated this way, it might seem trivial: in what other situation should a teacher correct a student? Let us put ourselves in the shoes of a student who is corrected only when they make a mistake. They write in their notebook $7 \times 8 = 65$. The teacher comes by and says “no, that’s wrong, $7 \times 8 = 56$”. Perhaps the student tries to present an argument in favor of their answer, but they soon realize that—in fact—the teacher is right, 7×8 is equal to 56. Another day, they write in their notebook $9 \times 11 = 99$: the teacher passes by and, after complimenting, moves on.

Another day, the student writes $25 \times 25 = 225$. The teacher notices and exclaims “No, look, you’re mistaken! $25 \times 25 = 625$”. And so it goes, day after day. Whenever the student gets it right, the teacher compliments; but every time the student makes an error, the teacher corrects them. What implicit clause do we expect the student to add to their Didactic Contract (as in Brousseau 1997): *Every time the teacher corrects me, I’m making a mistake.* What does this clause imply about the student’s internal dynamics? *If the teacher tells me I’m wrong there is no need to see the reason why, since I’m definitely wrong.* In other words, there is no longer an active role in learning where, if a mistake is made, one wants to understand where and why the mistake occurred. Instead, everything is left to the teacher’s positive or negative judgments. The dialogue is reduced to a black box (the teacher) that provides feedback and possible correction of one’s performance.

Consider the different dynamic that would be established if the teacher corrected the student not only when they were wrong, but—at times—even when they were right. After the student writes $7 \times 8 = 56$, the teacher will say it is wrong, that the result is a different number, and the student will argue in favor of their answer until the teacher is convinced. At this point, every time the teacher corrects the student, the student will no longer know if the teacher is "joking" (playing the role of a knave) or if they are telling the truth. As a result, during each interaction, the student will inevitably have to reflect on what they have said and try to argue their point.

This kind of interaction brings with it a logical benefit, as the student is expected to clarify all the steps taken to counter the teacher's arguments, even when they are right, thereby increasing their understanding of the statement. Simultaneously, it avoids a detriment at the emotional level: in a classroom environment where the true-false duality is not deeply explored, students risk associating the emotional reactions tied to mistakes with falsehood. As already discussed, if not properly addressed, false statements run the risk—in the student's perception—of being seen as mistakes and thus "as synonymous with failure and therefore to be avoided"(Zan and Di Martino 2017). This limits the possibility of a dialogical learning experience for the student, based on the contrast between true and false. Moreover, following this dynamic, the student becomes accustomed to arguing even in extracurricular situations, where everyone knows that being corrected does not necessarily mean they are wrong. The words of (Fossa 2019, p. 92) also provide an interesting insight: "the teacher is neither policeman nor judge, but gadfly".

In the classroom, students are rarely allowed to roam freely through the world of mathematics. Such roaming entails trying and failing (the word 'error' is derived from the Latin word *errare*, to wander or stray), and then modifying their approach on the basis of the information acquired. We believe that the character of the knave—a character with whom students generally sympathise—can also have a positive emotional return on mistakes. When exploring primary school teachers' opinions on logic, (Bibby 2002) found the majority believe "the objectivity of logic contrasts with the apparent subjectivity of the creative process", viewing logic as an obstacle to mathematical discovery. This belief seems to be based on a limited view of logic, in which logic is reduced to a syntactic formalism without semantic value and, above all, is considered a technique solely related to deduction, "assumed as an unproblematic foundation for the justification of knowledge" (Ernest 1991, p. 6). We believe that logic has a broader scope and can aid in the art of discovery.

6.4. A Logic Education

This leads us to reflect on the development of logical-mathematical skills in schools. Logic lies at the heart of mathematical and scientific thinking, and is fundamentally linked to certain elements of language. According to Ferrari and Gerla (2015), the continuous attention paid to mathematical language, to the distinction between language and metalanguage, and to the notion of interpretation when working with logic makes it a tool suitable for teaching and learning at every educational stage. However, as pointed out in (Durand-Guerrier, Boero, Douek, et al. 2012) and in (Mazzitelli and Millan Gasca 2022) the educational role of logic is not always recognized. There may be several reasons for this: on the one hand, formal logic can be seen as an unnecessary tool that risks complicating teaching practice; on the other, some believe that basic logical abilities are developed irrespective of a targeted theoretical treatment, as discussed earlier. For example, the concept of 'not' is "considered as a very simple notion [...] that does not need to be taught or discussed at this [primary school] level" (Durand-Guerrier 2021). But, according to the Author, the lack of an explicit treatment creates difficulties in understanding negation that can persist until university level, such as the fact that the link between the negation of a universal statement and the role of counterexamples is never fully clarified. Another reason for the omission of logic in the school curriculum is that teaching of mathematics tends to prioritize exercises that develop the skills required for improving grades and meeting school targets; hence, any topics that are not deemed essential for future study—even if considered important for individual development—are left to one side. The mistrust towards logic is clearly highlighted when dealing with symbolism. Returning to (Durand-Guerrier 2021), neglecting the links between logic and language leads to the paradox that mathematical formalism, which should serve to clarify concepts, becomes an obstacle to students' learning. Indeed, logical formalism is often only encountered when it is needed to express a mathematical concept not related to logic, and is viewed more as a syntactic abbreviation than a semantic clarifier. This is illustrated by the fact that a student may encounter quantifiers for the first time in the limit formula—a formula featuring three quantifiers as well as an implication—simply because it is no longer possible to express the concept in words. Introducing formal symbols for quantifiers this late in the game feels like a missed opportunity, akin to introducing the equality symbol for the first time when dealing with equations. Introducing a symbol to denote a concept requires a societal agreement on its meaning, and allows us to become aware that symbols are related to the context of use (Ferrari 2002). For instance, the logical conjunction AND will not capture every 'and' used in natural language; however, knowing how to recognise the differences and similarities in each case and context is an excellent starting point for learning the conjunction itself. Coppola, Mollo, and Pacelli (2019) provide an interesting analysis of the relationship between language, as an object to manipulate and reflect upon, and the development of logical abilities, considering specific scenarios of social interaction among primary school children (8–9 years old). A child is asked to behave like a robot that only obeys certain commands; the game therefore encourages the children to construct a simple symbolic language in which each symbol represents an instruction for the robot. These symbols do not correspond to those of standard logic, but the key point is to view logic as an "explicit expression of some aspects regarding language" (Coppola, Mollo, and Pacelli 2019). Moreover, the children can discover rules to "manipulate" the symbols of the created language (for example, rules that allow them to establish whether two different sequences of symbols can be considered equivalent in some way). We believe that logic, even in its formal and symbolic form, supports the development of rational thought and that it is therefore appropriate to dedicate time and space to logic and its symbols from primary education onwards. In our opinion, the idea that an understanding of basic logical concepts can be acquired automatically through standard mathematical teaching is wishful thinking.

As far as quantifiers are concerned, many authors emphasize the difficulty of working with them, once they begin to appear in mathematics education, and their relationship with everyday language is not always clear, sometimes resulting in a barrier to learning the quantifiers themselves. Most authors refer to secondary school level or university—that is, the point at which difficulties in reasoning (Piatek-Jimenez

2010) or in working with statements involving multiple quantifiers or alternate quantifiers (Epp 1999) become evident. Even though quantifiers, more or less explicitly, are present from the early years of mathematical education, the classical school practice does not recognize the need to explore them in an environment that is independent from their context of use. Think of the equilateral triangle that has *all* equal sides, the isosceles triangle that has *at least* two sides equal, even numbers, whereby there *exists* a number that when added with itself gives the even number, square numbers, whereby there exists a number that when multiplied with itself gives the square number, m greater than n means that there exists a k whereby $n+k=m$. Dubinsky and Yiparaki (2000) studied students' interpretations of statements involving both universal and existential quantifiers linking these to everyday discourse. They found that students do not have a strong understanding of quantifiers in everyday language, particularly concerning statements in which the existential quantifier precedes the universal quantifier. The two authors believe that students interpret statements containing quantifiers subjectively, in a personal context that they believe is implicit and shared with the interlocutor, and therefore it is even better to avoid situations familiar to students and focus on the syntactic aspect of the statement. Bardelle (2013) carried out a study with about three hundred Italian science undergraduates concerning the negation of quantifiers, showing that everyday communication heavily affects the interpretation of a variety of statements. Indeed, in common language, the meaning of some syntactic writings is often very different from the meaning attributed to those same writings by logic.

For instance, imagine the following dialogue:
A: “Hi, how are you?”
B: “Not so good, I have a cold”
A: “Oh my, *everyone* has a cold at the moment!”

Before proceeding, the reader is invited to stop and give a set theoretic meaning to that "everyone". What does the speaker A mean by saying “everyone has a cold”? They certainly do not mean that all human beings have a cold (as a purely logical interpretation would suggest), but nor do they mean that most human beings have a cold. They simply mean that a greater number of people than usual have a cold, a meaning totally different from the logical one. Also consider how these quantifiers, with their ambiguous interpretations, intervene in the construction of a sentence. For example, in Italian, a double negative such as “non so niente” (literally “I don't know nothing”) is considered correct, while the same sentence is preferably avoided in English. In French, the phrase “Aujourd 'hui, tous le bus ne circulent pas” (“Today, all buses do not run”) has caused confusion in real-life circumstances, given the possible interpretation of “Some buses run” rather than the intended “Today, no bus is running” (Durand-Guerrier 2020). Note that in Italian, negations can be singular or double depending on the order in which a concept is expressed: for example, “nessuno ha parlato” (“no one has spoken”) is equivalent to “non ha parlato nessuno” (literally, “not spoken has no one”), and "mai ci avrei pensato" (“never would I have thought of it”) is equivalent to "non ci avrei mai pensato" (literally, “I would not have never thought of it"). It is important to note the interesting case of Latin, where the formulation is not dissimilar to that of logical formalism. "Nemo non haec dixit" (literally "no one does not this say") actually means "everyone has said this", whereas "non nemo haec dixit" (literally, "not no one has said this") literally means "someone has said this"; "numquam non mendacia dixit" "has always lied", "non numquam mendacium dixit" "has sometimes lied"[4].

Khemlani, Orenes, and Johnson-Laird (2012) observe the difficulty of forming a mental model negation. As a consequence, people tend to assign a "small scope" to its meaning, that is, they tend not to consider all possible cases of negation of a given statement. Moreover, the authors state that the very symbol of negation can help to form a mental model of it. This is in line with our observations, which we will discuss in future sections.

The potential conflict between the mechanisms of interpretation of symbolic mathematical notations and those of everyday language is already present at primary school level (Ferrari 2021); consider not only the previously discussed "everyone" used with various meanings, but also the implication "if-then", which in natural language is often interpreted as an "if and only if". This does not mean that we must avoid the link to everyday language; on the contrary, our hypothesis is that the explicit study of quantifiers should start in connection with natural language from primary school itself. The aim of this is not to "correct" the ambiguities and underlying meanings of natural language, but to make students more aware and prevent the ambiguity of language from becoming a barrier to the understanding of mathematical statements. And indeed, most of the authors mentioned above identify early introduction and an explanation of the logic underlying the quantifiers as a possible solution to the problems encountered (Piatek-Jimenez 2010). We believe that expertise in formal logic will help interpretation of "informal" logical statements common in natural language. In our approach, the gradual introduction of symbols that express quantifiers and negation aims to distinguish common language from logical language, analyzing their similarities, differences, and ambiguities.

A. Selden and J. Selden (2007, p.11) highlight the difficulty related to the "inability to unpack the logical structure of informally stated theorems": when asked to recognize the logical structure of four syntactically correct statements informally phrased, two true and two false, university mathematics students—many in their third or fourth year—succeeded only 8.5% of the time. A possible cause of this phenomenon is that logical symbols of quantifiers and connectives may seem unfamiliar to students, as they are not addressed directly or encountered too late in a student's academic journey.

As A. Selden and J. Selden continue, being able to decompose the logical structure of informally stated theorems is important because the logical structure of a mathematical statement is closely linked to the overall structure of its proof. We emphasize that the fact that the logical structure of a formula is closely linked to its proof is

[4] https://accademiadellacrusca.it/it/consulenza/sulla-costruzione-della-frasenegativa-in-italiano 169

certainly true, as demonstrated in all the previous chapters. Conversely, the ability to decompose the logical structure of a theorem's statement allows one to know if an argument proves that statement. For instance, eight average-level university students in mathematics and mathematics education were asked to evaluate the correctness of "proofs" generated by students for a single theorem. After finding a proof of the converse particularly easy to follow, four initially erroneously stated that it was a proof of the original statement, and two of these maintained this opinion throughout the interview (A. Selden and J. Selden 2003).

We conclude by noting how A. Selden and J. Selden (2015) argue that the lack of proof-related skills in students at advanced levels can be attributed to a lack of argumentative practice in their schooling. They advocate that encouraging primary school children to reflect on their actions and provide reasoned arguments could be a precursor to the concept of proof. We add that it is important to focus not just on arguments in general, but specifically on those that pave the way for proofs, as discussed in the previous section.

6.4.1. Logic and Language in a Multilingual Context

The Zermelo education path, detailed in the next chapter, has been experimented with in multilingual classrooms in Italy and France. In such a setting, it is clear that additional difficulties arise other to the challenges with quantifiers and their negations.

In a multilingual context, the links between logic and language are connected to the fact that the structure of language can affect the thought process (Edmonds-Wathen, Trinick, and Durand-Guerrier 2016) and the knowledge of logical connectives in the language of instruction seems to be the most important variable in deductive reasoning for bilinguals from different countries (Dawe 1983). Ye and Czarnocha (2012) confirm the impact of natural language on the mathematical understanding of negation and identify a source of misconception initiated from incorrect French-English translation when working with non English students.

According to Meyer and Prediger (2012) mathematics lessons have high language requirements because learners need to understand, speak, and write many languages: everyday, educational and technical language. This is a challenge for every student, and in particular for non native learners (in the case of the authors, German is the language of instruction). According to the authors, the solution to these difficulties is often found in a “defensive attitude”, which tends to lower the language requirements. The solution should instead be more of a “attack”, that is, to encourage the learner to cope with linguistic challenges. After all, as we have seen, even in contexts where there are no apparent language problems, many authors argue for the need to make explicit the logical rules in order to foster students' comprehension and their ability to reframe statements that contain connectives, quantifiers, and negation (Epp 1999). According to Durand-Guerrier (2021), in a multilingual context, predicate logic can be used to unpack the logic of a given statement by identifying the logical categories, connectives, quantifiers, and their respective scopes. These concepts often remain implicit or are hidden through linguistic means depending on the language, and this can lead to ambiguities. Durand-Guerrier (2020) compares the understanding of the negation of quantifiers in different contexts, particularly in France and Tunisia—where lessons are taught in French at the secondary school level—showing some features of negation in French likely to introduce ambiguities or misunderstandings in class. "We might anticipate that such ambiguities inherent to the French grammar could be source of difficulties for non-francophone natives studying mathematics in French, and this especially as teachers are generally not aware of this" (Durand-Guerrier 2020, p.34). Durand-Guerrier reports on a study that compares the differing grammatical structures between Arabic, French and predicate calculus. According to the author, French and Arabic are not congruent for what concerns the negation of universal statements, while Arabic is congruent with predicate calculus. Indeed, in Arabic, when the negation is on the predicate, the scope of the negation is the predicate, not the sentence. The experimental results show that for most students, French universal statements with negation on the predicate were not interpreted as the negation of the sentence, in line with the standard interpretation in Arabic and, as already said, in logic. However, this does not seem to be an advantage for Tunisian students. More generally, the difficulties can be related to various factors:

- The study showed that nobody addressed this issue: neither the language teachers (Arabic or French), nor the mathematics teachers.
- The variety of languages used, as discussed by Prediger.
- The persistence of some differences in translation for certain quantifiers (such as each, every, either, neither, much, many, few, little) (Alabaqami 2020)
- The difficulty of not working in the mother tongue (Schwartz and Sprouse 1996).
- Social extraction and the family environment

6.4.2. Games of Logic and Computer Games in Mathematics Education

To conclude this chapter, it is essential to emphasize how our educational paths extensively utilize games, understood in various ways. On one hand, each teaching path is accompanied by an online game that explores the same topics, but from a different perspective. On the other hand, two-player games become the main focus of the final path, namely the Lovleis path. Therefore, it is important to briefly address the relationship between logic and games in the context of mathematical education.

One of the first games explicitly dedicated to logic is the classic board game *Game of Logic* by Lewis Carroll (Carroll 1887), designed to learn to solve syllogisms, *i.e.* statements containing quantifiers with one or more intermediate terms in common. The player has tokens of two colors at their disposal, which stand for *exists* or *none*, that they place on the diagram to indicate the existence or absence of elements in certain sets: essentially, it is a matter of interpreting appropriate set inclusions giving them logical meaning. Later, Carroll made his game more complex and advanced with *Symbolic Logic*, building a tool for the resolution of syllogisms (Carroll 1958). The fact that the author used the pseudonym Carroll, which he chose for his famous children's books, suggests that he wanted to emphasize the playful aspect of it. We also recall *Tarski's World* (Barwise and Etchemendy 1993), a computer-based introduction to first-order logic. The computer program introduces the semantics of logic through games in which three-dimensional worlds are populated with various geometric figures that are used by the player to test the truth or falsehood of first-order logic sentences. Dubinsky and Yiparaki (2000) discuss the use of quantifier games as a pedagogical tool to help students understand statements with alternated quantifiers: two players work on a sentence containing a universal and an existential quantifier, choosing values of two variables in two given sets and trying to verify a given relation among them. So, given some x chosen by player A, player B looks for some y such that a certain relation $R(x, y)$ holds. When speaking of logical games, we often find reference to puzzles or brain teasers, not really linked to formal logic or mathematics, although this does not mean that they cannot have an educational and logical value. For instance, Bottino and Ott (2006) analyze the use of computer mind games to develop strategic and reasoning abilities in primary school students.

The role that computer games can have in education, and particularly in mathematics education, is increasingly studied by researchers. In 2015 the *International Journal of Serious Games* dedicated a special issue to mathematics education (Vol. 2 n. 4). In the Editorial, Kiili, Devlin, and Multisilta (2015) identify some important characteristics for mathematics learning games, namely that these be founded on theoretically sound principles, integrate mathematics directly into the gameplay, rely on good pedagogical practices, and truly utilize the possibilities that game technologies provide for learning. In the same year the book *Digital Games and Mathematics Learning* (Lowrie and Jorgensen 2015) explored the influence and impact of digital games on young students' mathematics engagement, particularly focusing on learning situations beyond classrooms. The *Handbook of Research on International Approaches and Practices for Gamifying Mathematics* (Huertas-Abril, Fernández-Ahumada, and Adamuz-Povedano 2022) investigates the great challenge consisting in the design of materials for mathematics content learning, and the potential of game-based learning as a dynamic way to engage and motivate learners. It also addresses the possible aid that a computer game can provide in bilingual and plurilingual contexts.

Our stance is clear, regarding the need for a Logic Education starting from the early years of primary school. We believe it is essential for students to be introduced early on to the fundamental mathematical rules of proof and critical reasoning. To this end, as extensively discussed, it is **indispensable** to develop specific knowledge about truth values, quantifiers, and connectives (exploring their Dialogue Rules), as well as to delve into games and strategies.

The following three chapters will be devoted to a detailed analysis of three possible logic educational paths. The game $\mathbf{T}\mathscr{UVA}$ served as a foundation, guiding the conception of the general structure of the paths from the very beginning and, subsequently, inspiring many specific activities. Starting from the study of sets and quantifiers with

Zermelo's path, we will continue by examining language and connectives through Bul's path. The last educational path, named Lovleis, offers an analysis—in a fairy-tale context—of two-player games, with the goal of delving deeply into the concept of a strategy.

7. Zermelo Educational Path

In this chapter, we describe and analyze the educational path of *Zermelo*[1].

Through the description of various sets drawn on boards—containing numbers, figures, animals, etc.—the path Zermelo aims to develop the students' sense of observation, as well as their ability to express and verify properties of elements of certain sets. Correct and incorrect descriptions are proposed and requested, with the aid of Smullyan's characters: the knight who always tells the truth, and the knave who always lies (Smullyan 1987). The activity leads to the use and analysis of the words *all*, *at most*, *at least*, and *none*, as well as their negation.

7.1. Background and related work

In the 1960s and 70s, during the period of the so-called *Mathematique Moderne*, in France and Belgium, as in other countries, logic and naive set theory were introduced into pre-university mathematics curricula. This phenomenon, which in the United States was called *New Math* (with slightly different contents), was at the center of a heated debate with strongly critical interventions by some mathematicians (Kline 1973). The *modern mathematics* movement originated with the Bourbakist movement, and its most important moment was the Royaumont Conference near Paris (1959), sponsored by OECD on the theme *New Mathematics*. On that occasion, the Bourbakist mathematician Jean Dieudonné issued the famous cry: *A bas Euclid*, dramatically conveying the need to go beyond traditional teaching. In his address, Dieudonné proposed, among other things, to introduce the language and symbolism of set theory from primary school onwards. The Bourbakist project intended to frame all mathematics in set theory, trying to reconcile logical genesis with the psychological genesis of the individual (Piaget 1955; Piaget, Beth, Dieudonné, et al. 1955). The Bourbakist reforms took root in many countries and at various educational levels (De Bock 2023). As for primary schools, an example of the Bourbakist influence is the idea, which gained ground among some teachers and in many textbooks, that numbers should be introduced starting from sets. This idea is based on Cantor's definition of an ordinal number as an equivalence class of well-ordered sets with respect to the isomorphism relation. Thus, in primary school, set theory was used as a means for a new approach to arithmetic, using the 1-1 correspondence to introduce the cardinal concept of number, seeing order of numbers as an inclusion of sets, their sum as the union of disjoint sets, and multiplication as a Cartesian product (as proposed also by the English Nuffield Project). The pedagogical limitations of this approach are

[1] The name is a tribute to Ernst Zermelo.

highlighted by M. Pellerey (1989), who stresses the importance of *mathematizing raw situations*, postponing the introduction of more sophisticated mathematical structures until a child reaches a greater level of awareness. Despite various criticisms, elements of set theory remained in the primary schools of many countries for a long time. For years, textbooks for primary school included an early chapter about sets and their intersection, union, and 1-1 correspondence. Regarding Italy, the overcoming of the Bourbakist experience was highlighted in the 1985 elementary school programs, which continued to assign considerable importance to the theme *Logic*, but this was seen as a cross-curricular skill and not as mathematical content to be added to those traditionally taught in elementary schools. *Concrete* activities were suggested, rich in *logical potentialities: classifications through attributes, inclusions, serializations.* The problem of the symbolic approach was explicitly addressed, but with particular attention to awareness and the meanings to be attributed to symbolic representations. Formal symbolization of logical-set operations was not considered necessary, preliminarily, for the introduction of natural numbers and arithmetic operations. These programs are no longer in force. For an in-depth examination of the issue, especially from an Italian perspective, please refer to (Veredice 2023a).

Our proposed path, *Zermelo*, is far from the Bourbakist view and shares some points with the Italian programs of 1985 as well as with the present programs (Indicazioni Nazionali) for primary schools. Zermelo has language, description, verification of true or false properties and quantifiers as its main topics. We do not use set theory from a foundational point of view but from a cognitive one, placing the focus on the student and not on mathematical formalization, relegating to symbolism the role of sharing meanings rather than creating them.

The sets used in the course are concrete entities, useful for creating logical reasoning on them. Each time, reasoning is based on a single set, seen as an environment in which to verify certain statements. There is absolutely no foundational claim for other areas of mathematics. In particular, we are totally distant from the idea of introducing the number as an equivalence class among sets and the arithmetic operations consequently.

In other words, we give to set theory and logic, in addition to a crucial role for the cognitive development of the student, a role for analyzing the mathematical knowledge that one *already* possesses. A similar role is played by grammar in the study of a language. Hoping to teach a student to speak starting from the study of grammar would seem foolish to anyone with common sense.

7.2. Description of the path

The *Zermelo* educational path has been proposed and experimented with in primary schools and—*mutatis mutandis*—also in high schools, in Italy and France.

7.2.1. First Activity: Knights and Knaves

In the first activity, the characters of the knave and the knight are introduced. The knave character always lies, while the knight always tells the truth. Depending on the school grade, these characters are introduced in different ways. In the early years of primary school, masks of the respective characters were given out to be colored and cut out, while in the later years they were distributed already made, and in lower secondary school sometimes it was preferred not to use the masks at all. However, having overcome the embarrassment of having to use masks with somewhat older students, we believe the mask artifact is also useful for grades beyond primary school.

A first activity that is always carried out is to ask to formulate sentences—more properly, statements—interpreting the role of the knight or the knave at will. This phase is extremely important because it offers the opportunity to focus on a fundamental question: *How should these statements be formulated?*. First of all, the statements must be meaningful, in other words, neither syntactically incorrect constructions like "$3+++==$" nor syntactically correct constructions to which it is not possible to assign a truth value, like "The yellow umbrella" are acceptable. However, no student, when called to play the role of the knave and the knight, has ever made mistakes of this kind, at any school grade. Secondly, the statements must be, in some way, objective: that is, they should not present personal tastes or opinions to which it is difficult to assign a truth value.

In this case, it has happened, a few sporadic times, that students would say phrases like "volleyball is beautiful", and it is interesting to make the class reflect—when such cases arise—on the subjectivity of the statement. Another important characteristic that a statement must have is to be *verifiable*. Even with very young students, it is possible to start talking about the verifiability of a statement, asking whether it is possible for the interlocutor to assign a truth or falsehood value to a given statement. For example, it has happened to hear phrases like "my sister's name is Maria": at that point, it is then appropriate to ask "How can you be sure that her name is Maria?". It then turns out that some students support their classmate's statement and try to convince the experimenter that indeed, yes, the sister is named Maria. Another step to take, introducing the characters of the knight and the knave, at appropriate school levels, is to ask that statements be made in a mathematical context, of an arithmetic type—like "$2+2$ equals 4" said by a knight—or even geometric—like "a triangle has 8 sides", clearly said by a knave. Although the teacher's initial examples were of incorrect calculations, such as “2 plus 3 is equal to 2”, some students opted for diverse examples of mathematical falsehoods, such as “100 has two figures”.

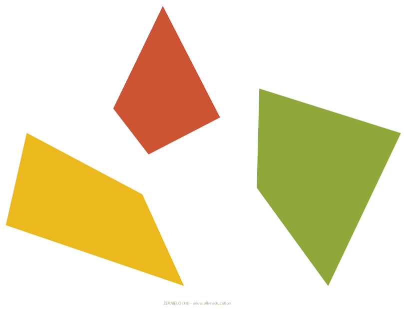
Introducing the characters of the knave and the knight, another activity is also proposed, in which the student must understand whether the sentence spoken by a classmate or the teacher was said by a knave or a knight.

This introduction is shared by the educational path Bul 8, but in Zermelo the focus is shifted to *description*. In fact, we then start by describing the objects of the environment where the lesson is taking place, interpreting the two roles, so "this chair is brown", "this notebook is round". And, in a similar way, one must understand whether the person making the description is a knight or a knave.

7.2.2. Second Activity: Sets

The second activity involves the introduction of the main artifact of the Zermelo course: the tables. The Zermelo tables are drawings that can be projected onto an interactive whiteboard or printed, depicting sets: sets of people, animals, figures, numbers, etc., as shown in Figure 7.1.

(a) A set of people.



(b) A set of geometric shapes.

Figure 7.1.: Examples of Zermelo tables.

In this activity too, the fundamental activities are two: describing what is being observed, interpreting as the knave or the knight, and understanding whether the speaker, listening to a description made by a classmate or the teacher, is a knave or a knight.

The description activity is proposed in an absolutely free manner: each student can make the descriptions they find most appropriate, no matter how simple they may seem. How then does the teacher guide towards the goals they have set? With the descriptions made by the teacher themselves. That is, it is hoped—and in fact obtained, in reference to the experiments carried out—that the students begin to emulate the descriptions made by the teacher. In this way, creative freedom is left, where the teacher themselves can pick up on interesting examples made by the children, but where the opposite can also happen in an unforced manner. This way, the teacher can also steer the class towards the subsequent activity on quantifiers.

7.2.3. Third Activity: Quantifiers

The third activity continues with the foundational theme of the course: quantifiers. Before delving into certain aspects of the activity, it's vital to make a clear distinction: although quantifiers are often discussed in primary school, they are addressed in a linguistic context rather than a purely formal and mathematical one. This can lead to confusion between mathematical quantifiers and the less precise quantifiers of natural language such as 'few' and 'many'. While 'few' and 'many' certainly play a role in a student's cognitive development and their exploration is significant, it is essential to differentiate mathematical quantifiers—which have clear and precise meanings—from everyday language quantifiers, where meanings are more nuanced. The quantifiers referred to in this path are the classic ones: *all*, *at least*, *at most*, *none*.

Quantifiers are used to describe sets as in the previous activity, and from this point, the integration with the Zermelo Game—an online game described in the next section—begins. Various reflections can be proposed in class. The first is that—using negation—different words can express the same concept: saying 'all numbers are even' is equivalent to saying 'no number is odd'; in general, stating that no object has a certain property is the same as saying that all objects lack that property.

7.2.4. Fourth Activity: Symbols and Core Quantifiers

From the fourth activity onwards, the focus shifts exclusively to the two quantifiers typical of mathematical practice, through which all other quantifiers can be expressed: 'all' and 'at least one'. These quantifiers are also introduced through symbolism, using the usual symbols $\forall$ and $\exists$. In all our experiments, even with second-grade primary school classes, these symbols were never problematic; in fact, they always seemed to intrigue and pique curiosity (many teachers have reported that their students show this attitude towards any unconventional symbol, like Egyptian or Sumerian symbols). The narrative we propose for introducing these symbols is as follows: We explain that the knaves and knights, inhabitants of Smullyan's island, prefer to be succinct in their writing (a tactic that will also be useful in the Bul path) and use symbols instead of some words. This also helps them to be sure of what the speaker means, as sometimes the same word can have different meanings for different people. Specifically, the knaves and knights use the symbol $\forall$ to indicate 'all' and the symbol $\exists$ to indicate 'at least one'. The first thing we do is to ponder why these particular symbols were chosen, leading to the discovery that $\forall$ derives from the English word ALL and $\exists$ from the English EXISTS[2]. It's also important to emphasize the equivalence between the expression 'exists' and 'there is at least one'.

In the more advanced school grades, we will certainly discuss the historical introduction of symbols for quantifiers. Once the quantifiers are introduced, they can be used for written descriptions of tables (see Figure 7.2 and 7.3).

[2] In fact, it's plausible to assume it derives from the Italian 'Esiste', as we saw in Chapter 2.

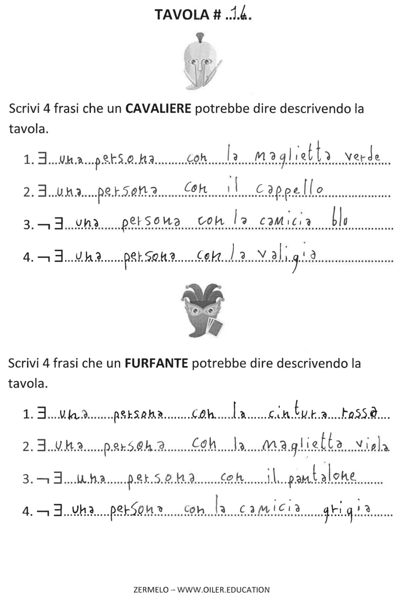


TAVOLA # 16

Scrivi 4 frasi che un **CAVALIERE** potrebbe dire descrivendo la tavola.

1. ∃ una persona con la maglietta verde
2. ∃ una persona con il cappello
3. ¬∃ una persona con la camicia blu
4. ¬∃ una persona con la valigia

Scrivi 4 frasi che un **FURFANTE** potrebbe dire descrivendo la tavola.

1. ∃ una persona con la cintura rossa
2. ∃ una persona con la maglietta viola
3. ¬∃ una persona con il pantalone
4. ¬∃ una persona con la camicia grigia

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Figure 7.2.: Description of a table using the symbol ∃.

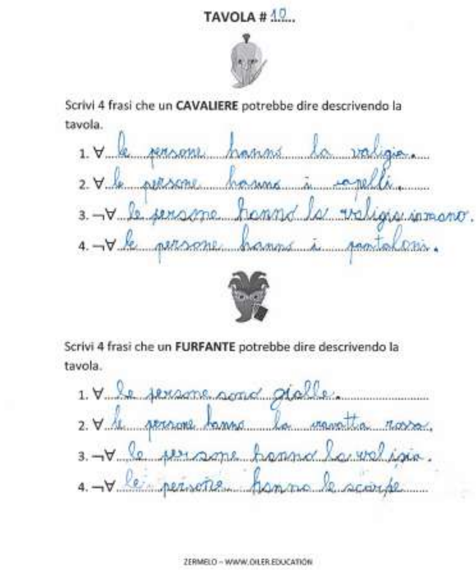
TAVOLA # 10

Scrivi 4 frasi che un **CAVALIERE** potrebbe dire descrivendo la tavola.

1. ∀ le persone hanno la valigia.
2. ∀ le persone hanno i capelli.
3. ¬∀ le persone hanno la valigia in mano.
4. ¬∀ le persone hanno i pantaloni.

Scrivi 4 frasi che un **FURFANTE** potrebbe dire descrivendo la tavola.

1. ∀ le persone sono gialle.
2. ∀ le persone hanno la cravatta rossa.
3. ¬∀ le persone hanno la valigia.
4. ¬∀ le persone hanno le scarpe

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Figure 7.3.: Description of a table using the symbol ∀. The use of the plural form is because the quantifier is read as 'all' and not as 'for each'.

Up to this point, attention to argumentation has been addressed only implicitly by the teacher. From now on, even with the aid of Zermelo Game—which will be analyzed in the following section—a heightened focus will be placed on argumentation and dialogue between two individuals holding opposing views on a statement containing quantifiers. The argumentative technique proposed in the classroom is the same as suggested by Game Semantics. Specifically, if a person makes a universal statement, someone seeking to refute it must find a counterexample, whereas if a person makes an existential statement, they themselves will be called upon to provide an example.

Referring to the Figure 7.4, the teacher says "for every figure in the table, that figure is a triangle", and the student has to correctly identifies this as false. The teacher will ask why, and the student will provide a counterexample to the teacher's statement, namely a figure that is not a triangle. Great care will be taken with the language used, striving to construct increasingly precise sentences.

Figure 7.4.: The counterexample to the statement "every figure is a triangle" is the red quadrilateral.

7.2.5. Fifth Activity: Union and Intersection

In the fifth activity, we introduce the concepts of union and intersection. As previously discussed, these concepts are not highly regarded among teachers and researchers due to their association with the "modern mathematics" movement. However, we believe that union and intersection are important concepts in the mathematical development of an individual and also serve as precursors to their corresponding logical connectives.

We then return to the tables to describe or interpret descriptions regarding the union and intersection of tables. Performing the union of two tables is quite straightforward, it simply involves placing them side by side and then describing their union. However, for intersection, an additional step is required: we choose two tables that have elements in common, find their intersection, and then describe it. We also believe it's important, with depth varying according to the educational level, to understand the relationship between union, intersection and quantifiers. Specifically, it is relevant to analyze in class that if a universal statement is made about the union of two tables, then this statement must also hold true for each of the tables. Conversely, an existential statement about the union does not necessarily apply to each table individually. On the other hand, a universal statement about the intersection of two tables may not be true for both sets, whereas an existential statement regarding the intersection will certainly hold true for both tables individually.

7.2.6. Sixth Activity: Subsets

The final topic addressed is subsets, a fundamental theme for introducing logical connectives as well. It is important to emphasize that, to identify a subset, we preferred to circle its elements one by one, rather than grouping them with a single closed line. This approach, as we will see, facilitates understanding and subsequent discussion.

The logical connectives to be explored, and the depth with which we treat them, vary according to the course level. To give an idea, we usually start with negation: students are asked to circle the elements of a table that possess a specific property. It will be noted that a circled element possesses the property, while an uncircled one does not. Continuing on the same table, other elements can be circled that have another property, allowing observation that elements circled twice possess both properties (connective $\wedge$), those circled at least once have at least one of the properties (connective $\vee$), and those not circled do not have either property.

The case of implication, along with the concepts of necessary and sufficient condition, is more complex, but equally stimulating. The basic idea is that, although it may be false that all elements of a table possess a certain property, if we limit ourselves to those that possess another property, then it becomes true. This discussion is clearly related to the one made in the introduction of Chapter 5.

7.3. Zermelo Game

The software *Zermelo Game*, accessible at `www.oiler.education/zermelo`, is a free online game designed to support educational activities related to sets and quantifiers[3]. The game has been used in experiments carried out in various contexts: from the beginning of primary school to the end of high school. Within primary school, *Zermelo Game* has been integrated with the *Zermelo* educational path described above. Also in the path carried out with high school students, the game is alternated with moments of work on quantifiers, argumentation and deduction. We will here describe the mathematical framework behind the software, how the software works and how it was used during classroom activities.

The game was designed by myself and developed by Jacopo Zuliani, Mattia Sanchioni, and Giulia Balboni employing classic online game elements to enhance student motivation and engagement in learning logic through captivating visuals, adjustable difficulty levels, self-competition and rankings, self-paced learning, and instant feedback upon errors.

[3] The game is available in English, French, and Italian.

7.3.1. Design and Implementation

As described above, the game is part of a educational path focused on quantifiers, which aims to develop four fundamental skills: evaluating sentences containing quantifiers, constructing sentences with quantifiers, building sets that respect certain conditions containing quantifiers, and determining the appropriate quantifier to apply to a property given a set of elements. *Zermelo Game* specifically focuses on the last skill, although the others are closely interconnected.

When playing the game, the main goal is indeed to determine whether all or not all elements of a given set enjoy a certain property, or alternatively, whether at least one or none of them do. The game presents various environments that may also require other mathematical skills. Seen from the student's side, the goal in the game is to earn as many points as possible within a set time by correctly answering the questions posed.

On the home page (Figure 7.5), you select which quantifiers to play with (you can also select both); the environment, which describes what types of objects will appear on the screen (colors, polygons, numbers, or bags); the level and the time available in the match. You can also choose the negation or witness modes, which we will see later.

Figure 7.5.: The Zermelo Game's homepage

In *Zermelo Game*, the use of symbols to indicate quantifiers is gradually emphasised. In earlier levels, symbols are accompanied by the equivalent expression in natural language, and the teacher and the player can ignore them. In later levels, understanding the symbols becomes important, even though the teacher will always invite students to consider the linguistic interpretation of the expressions they read. In the *bags* environment, the notation $Y(x)$ is also used to express that the object x has the property Y (*e.g.*, BLUE (ball)). This notation is introduced in the *Bul* education path (Oiler, 2021b), but will be easy for the teacher to explain it if the students do not know it.

The levels and gaming environments are intended to progressively develop competencies. The polygon environment requires competencies in the thematic core of geometry, the numbers environment in the thematic core of arithmetic, while the colors and bags environments refer exclusively to the thematic core of logic.

7.3.2. Colors

The color environment requires logical skills only, with no other mathematical skills involved. In the only available level, once the two quantifiers (*all* and *at least one*) have been chosen, players need to identify whether all the balls shown are of a certain color (red, green, blue) or if at least one or none of them are of that color (Figure 7.6).

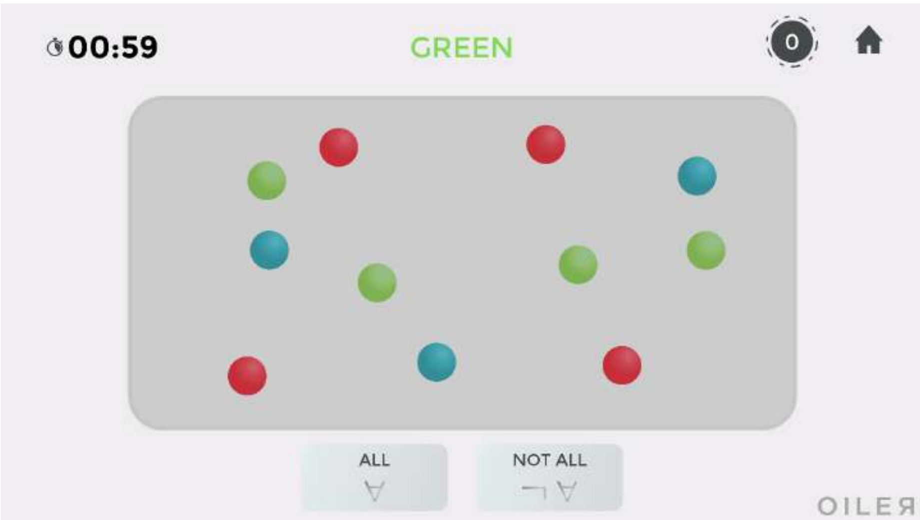


Figure 7.6.: Not all balls are green.

7.3.3. Polygons

In this environment, players need to recognise specific properties of polygons. The levels of difficulty are organised as follows:

In Level 1, the elements that appear are polygons (i.e., plane figures bounded by segments), and the properties are TRIANGLE, QUADRILATERAL, PENTAGON, HEXAGON (polygons with more than six sides do not appear). For variety, some polygons have shapes that are not typically used in school practice: to respond, it will always be sufficient to count the number of sides (or equivalently the angles) of the polygon in question (Figure 7.7). This level can be proposed from the first grade.

In Level 2, in addition to the properties of Level 1, the EQUILATERAL property is introduced. A polygon is said to be equilateral if all its sides are equal. Similarly, a polygon is not equilateral if it has at least one pair of different sides. We note that an equilateral quadrilateral is commonly called a rhombus (from the Greek rhómbos, meaning spinning top).

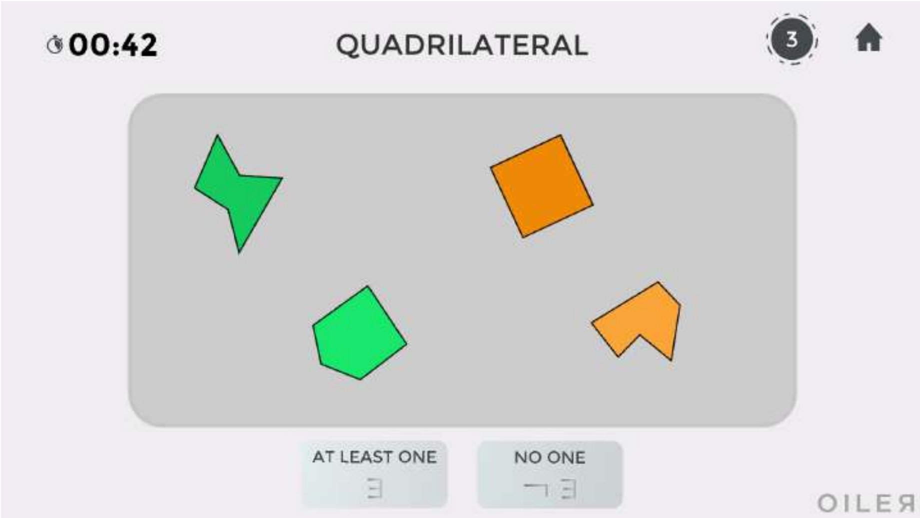


Figure 7.7.: Since there is an orange square, the answer is *AT LEAST ONE*.

In Level 3, in addition to the properties of Levels 1 and 2, the properties AT LEAST TWO EQUAL ANGLES, AT LEAST TWO EQUAL SIDES, ONE RIGHT ANGLE, ONE OBTUSE ANGLE are introduced. The properties "one right angle" and "one obtuse angle" are to be understood as "the polygon has at least one right/obtuse angle". In fact, very often, the expression "at least one" is implicit in language. The properties "at least two equal angles" and "at least two equal sides" appear exclusively in reference to triangles. The class will note that the properties are equivalent: a triangle has at least two equal sides if and only if it has at least two equal angles. A triangle of this type is commonly called isosceles (from the Greek *isoskelés*, where *ísos* means equal and *skélos* means side).

Starting from Level 4, the response buttons change slightly, giving more space to symbolism: in particular, the expression "all" is replaced exclusively by the symbol ∀ and the expression "at least one" is replaced exclusively by the symbol ∃. In this way, we encourage students to move away from writings toward symbolism.

In Level 4, in addition to the previous properties, the property AT LEAST TWO PARALLEL SIDES is introduced. For a triangle, it is impossible to have two parallel sides, while a quadrilateral with at least two parallel sides is usually called a trapezoid (from the Greek *trapézion* meaning small table). Finally, Level 5 adds the properties EQUIANGULAR and REGULAR. A polygon is called equiangular when it has all equal angles, and is called regular when it is both equiangular and equilateral. A quadrilateral with equal angles is commonly called a rectangle (because if a quadrilateral has all equal angles, it consequently has all right angles), while a regular quadrilateral (which is both a rectangle and a rhombus) is commonly called a square.

7.3.4. Numbers

In this environment, players need to identify specific numerical properties. The levels of difficulty are organised as follows:

In Level 1 of numbers, numbers between 0 and 9 appear and the properties EVEN and ODD are used. A number is even when there is a number that, when added to itself, results in the number itself. For example, 10 is even because $10 = 5 + 5$, while 0 is even because $0 = 0 + 0$. A number that is not even is odd (Figure 7.8). In Italian, as in several other languages, the word for odd *dispari* comes from a negation of even (*pari*), using the negation prefix "dis-".

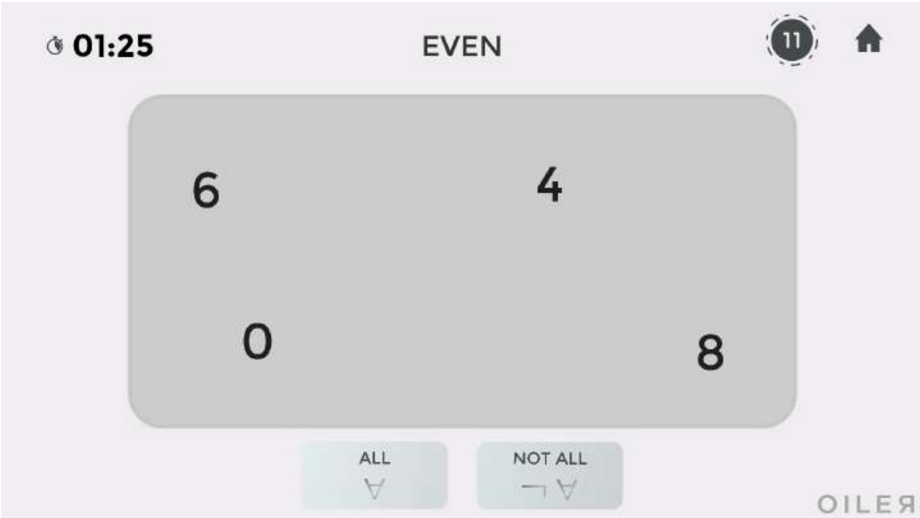


Figure 7.8.: The answer is *ALL*.

More precisely, referring to the notations introduced in the Zermelo path, a number n is even if $\exists x(n = x + x)$. As will be explained in more detail in the *Witness mode* section, when arguing with a $\exists$, a witness must always be provided. In particular, it is not enough to say "6 is even", but it is always necessary to specify why: 6 is even because $3 + 3$ equals 6.

In Level 2, alongside the properties in Level 1, we are introduced to GREATER THAN and LESS THAN. In this case, the properties extend to numbers between 0 and 100, as shown in figure 7.9.

In Level 3, in addition to the previous properties, the properties DIVISIBLE BY 3, DIVISIBLE BY 4, DIVISIBLE BY 5, and LAST DIGIT also appear. We note that divisibility is an existential statement: indeed, x divides y means $\exists k(k \times x = y)$. It is therefore appropriate to request a witness from the person stating divisibility.

7.3.5. Bags

In this environment, alternating quantifiers appear for the first time: the player must consider phrases like "all bags contain at least one blue ball" or "at least one bag has all blue balls"—that is, phrases with two quantifiers.

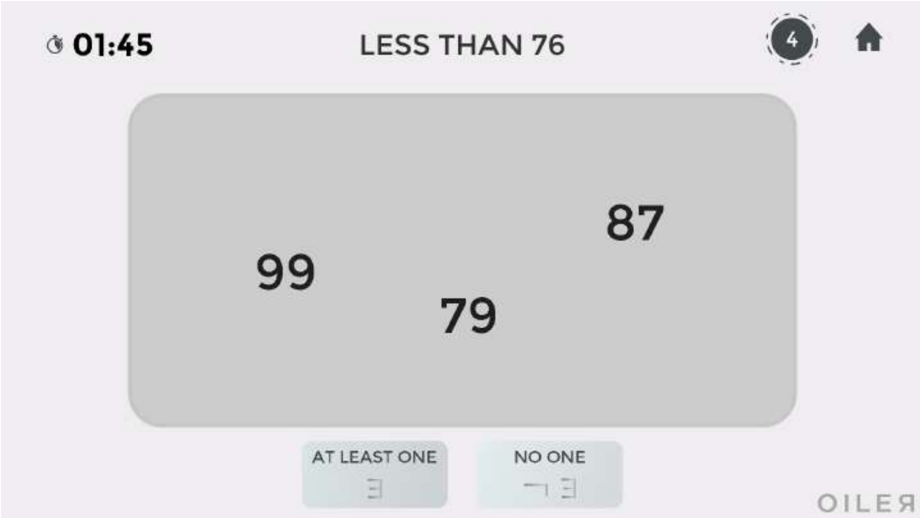


Figure 7.9.: The answer is *NO ONE*.

A property referring to individual bags appear at the top of the screen. At the bottom, it asks whether there is a bag with this property, that is, a bag that has all blue balls (Figure 7.10. The answer is negative and therefore you must click on the button "¬∃ BAG".

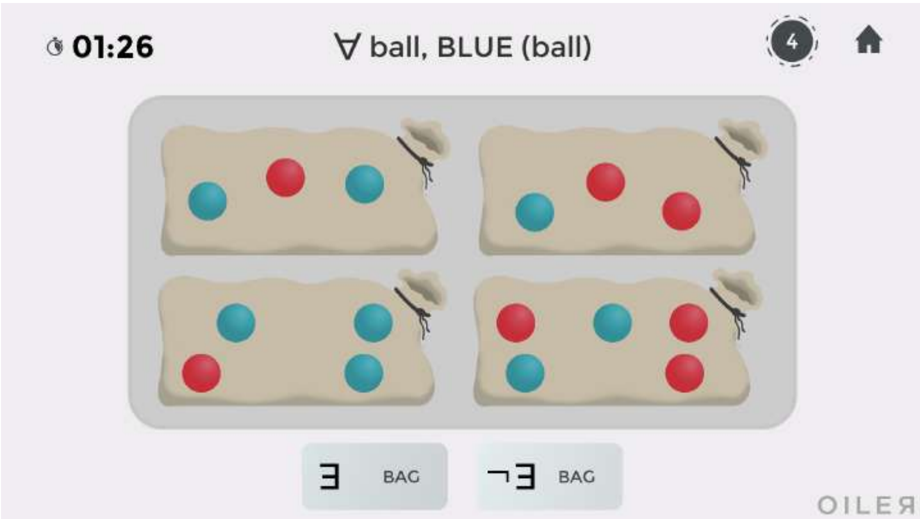


Figure 7.10.: The answer is ¬∃ *BAG*, because no bag has all blue balls in it.

As a second example, consider the following situation. At the top appears the property that refers to individual bags, ∃ ball, BLUE(ball), to be read as "there exists a blue ball" or "at least one ball is blue". At the bottom, it asks whether all or not all bags verify this property, that is, whether all bags contain at least one blue ball. The answer is negative and therefore you must click on the button "¬∀ BAG" (Figure 7.11).

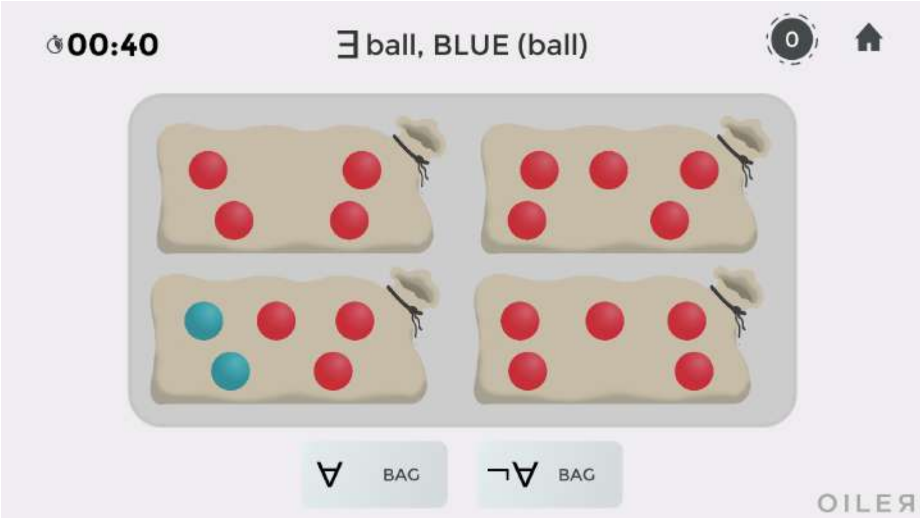


Figure 7.11.: The answer is ¬∀ *BAG*, because three bags have no blue ball in them.

7.3.6. Witness Mode

If you select *Witness Mode* on the home page, a new rule is added. When you click on "not all" or "at least one" (that is, when you give an existential answer), you must provide a witness, i.e., indicate an object that testifies to the choice made. In the case of "not all", you are required to click on an object that does not have the indicated property. For example, referring to the following figure, after clicking on "not all," you need to select a polygon that is not a pentagon, in this case the green rectangle (Figure 7.12).

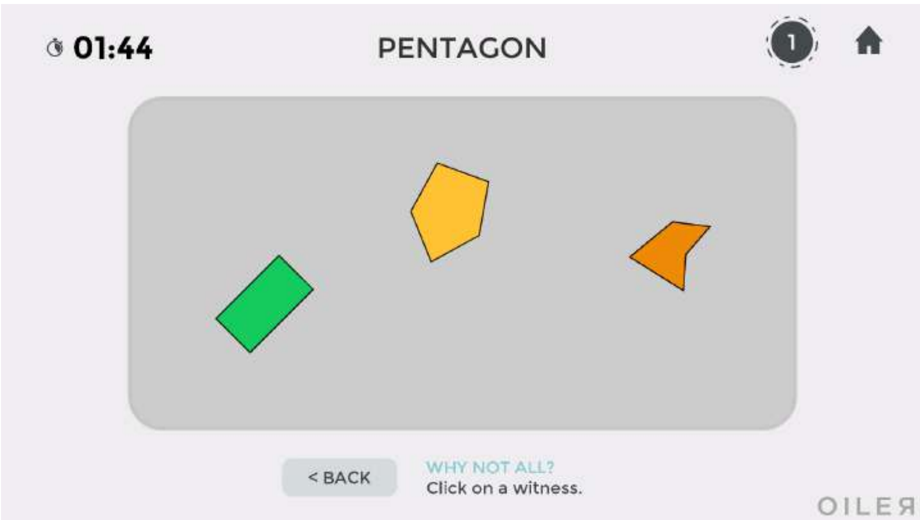


Figure 7.12.: The witness for the answer is the green rectangle.

Similarly, in the case of "at least one", you are required to click on an object that has the indicated property. For example, referring to the following figure, after clicking on "at least one", you must select an even number, in this case, 0 (Figure 7.13).

Figure 7.13.: The witness for the answer is 0.

The witness mode in the "bags" environment—while following the same principle presented above—is more complex: in this case, a real dialogue occurs between the player and the computer. To better understand the situation, let's use an example.

If you assert that there is at least one blue ball in all bags, then you will be able to identify a blue ball in any bag the computer may choose (Figure 7.14).

Here are some examples of possible scenarios:

- if you claim that all bags have all red balls, you don't have to do anything (0 total clicks);
- if you claim that not all bags have all red balls, you need to click on a bag where at least one ball is blue, then click on a blue ball (2 total clicks);
- if you claim that at least one bag has all red balls, you need to click on the bag with all red balls (1 total click, made on the bag);
- if you claim that all bags have at least one red ball, you need to click on a red ball in a bag chosen by the computer (1 total click, made on a ball).

7.3.7. Negation Mode and Its Interaction with Witness Mode

If you select the negation mode, then also the negation of the usual proprieties can appear at the top of the screen. The symbol ¬ is indeed read as "not". For example, being ¬ GREEN means "not being green": in the case of the game, this means being either red or blue. Similarly, being ¬ EVEN means "not being even" or odd. In the following situation, you are asked whether at least one ball is not green, therefore either red or blue (Figure 7.15), or not.

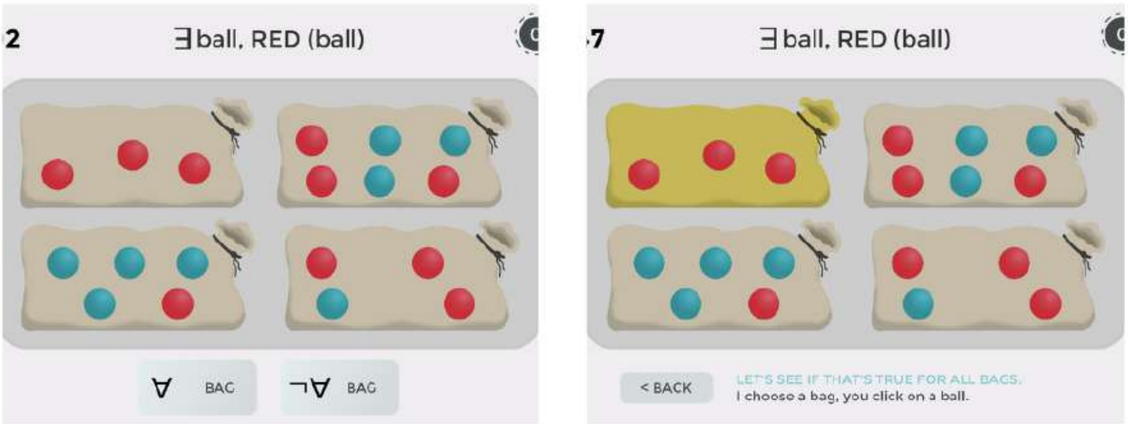


Figure 7.14.: The player states that all bags have at least one red ball. The environment chooses a bag and the player has to identify a red ball in it, *i.e.* a witness.

If you are also playing in witness mode, you will then proceed to click on a red or blue ball. It is worth highlighting the tricky case shown in the Figure 7.16.

Being a "non-hexagon" means, in the context of the game, being a triangle, quadrilateral or pentagon. In the Figure 7.16, therefore, not all shapes are non-hexagons, because the blue polygon at the bottom right has 6 sides. If you are playing in witness mode, you will have to click precisely on that hexagon.

7.4. Analysis of the Classroom Experience

In this section, we will explore some parts of trials run with students who used *Zermelo Game* within the path *Zermelo*. For the purpose of this analysis, we will focus primarily on the *colors* and *bags* environments, as these environments incorporate logical elements without also requiring auxiliary mathematics skills, as is the case for the polygons and numbers environments.

The trials were conducted both at primary school (Italian and French schools) and high school classes (Italian school). The observations reported here are based on footage and notes taken by the researchers involved. Primary school trials were carried out directly in the classroom, in the presence of the class teacher and a researcher. For the secondary school trials, groups of approximately 20 students attended sessions held at the university by one of the study researchers.

In the French primary schools the game was played in a collective way by projecting it on the interactive whiteboard. Students were then invited in turn to answer one of the questions. In the Italian primary schools the work was carried out in the classroom

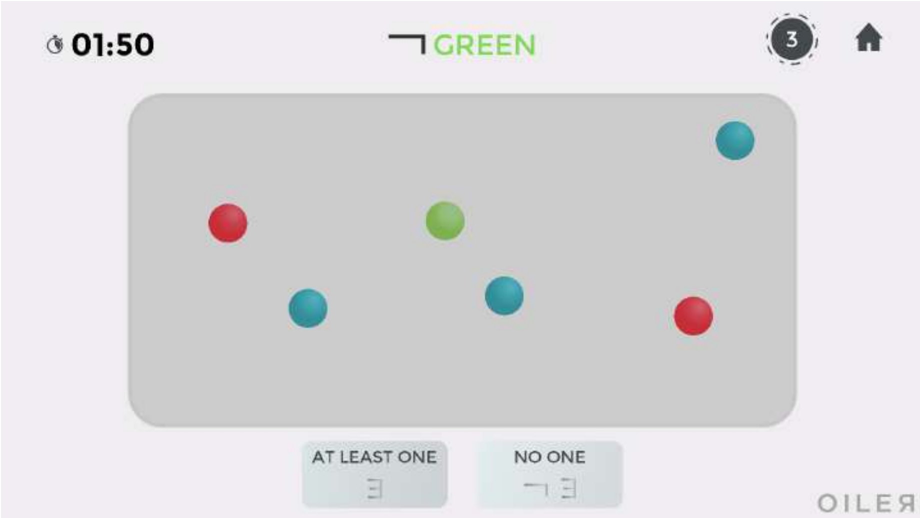


Figure 7.15.: The answer is that at least one ball is *not green*, because there is at least one red or blue ball.

playing as a group, as well as in the computer lab, in which students worked together in pairs. In third-year classes the students also competed against one another on the game rankings.

By contrast, high school worked independently, or in pairs, at the computer, and then explained to the researcher what they had done. They mostly worked within the *bags* environment, and competed against one another on the game rankings.

Overall, students enjoyed using the software and became comfortable with the game after a little practice. After further practice, we observed significant improvements in the students performance, as can be seen from the scores on the 1-minute leaderboard.

The leaderboard is readily accessible online, located directly beneath the game interface, and undergoes instantaneous updates after each gameplay. To gain access to the leaderboard, both quantifiers must be selected, as well as the 'witness' mode and the 'negation' mode, and the gameplay duration should be set to one minute.

The leaderboard has proved to be considerable effective in motivating and engaging the students, who enthusiastically embraced the competitive aspect it offered. The extremely high scores shown in Figure 7.17 serve not only as an indication of the significant level of engagement but also, as we will see later, as evidence of the awareness of the underlying mechanisms of argumentation.

7.4.1. Primary School

Zermelo Game was introduced gradually, with the difficulty increased bit by bit in each session. On average, *Zermelo Game* was introduced in the second activity with the class. The first game was played with the quantifier ALL and on Level 1 of the colors environment, with neither negation or witness mode. Students were asked to

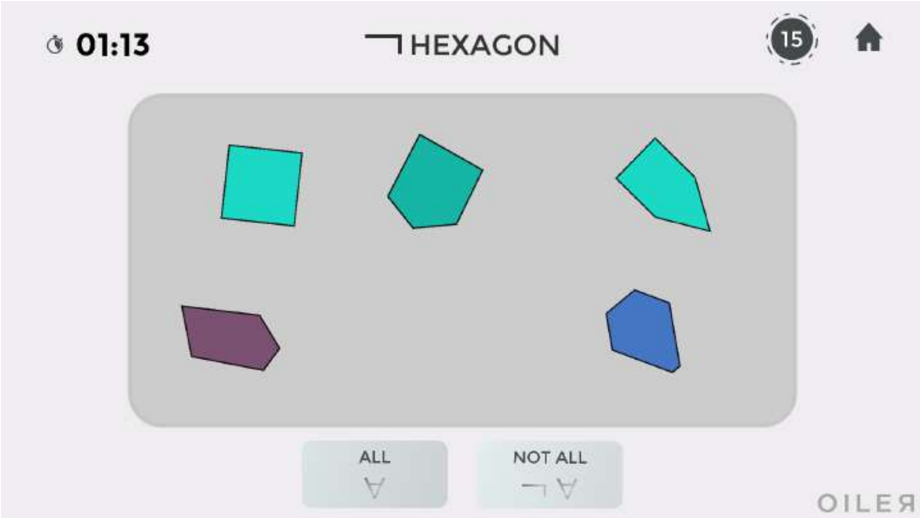


Figure 7.16.: The answer is *NOT ALL* because at least one polygons is an hexagon.

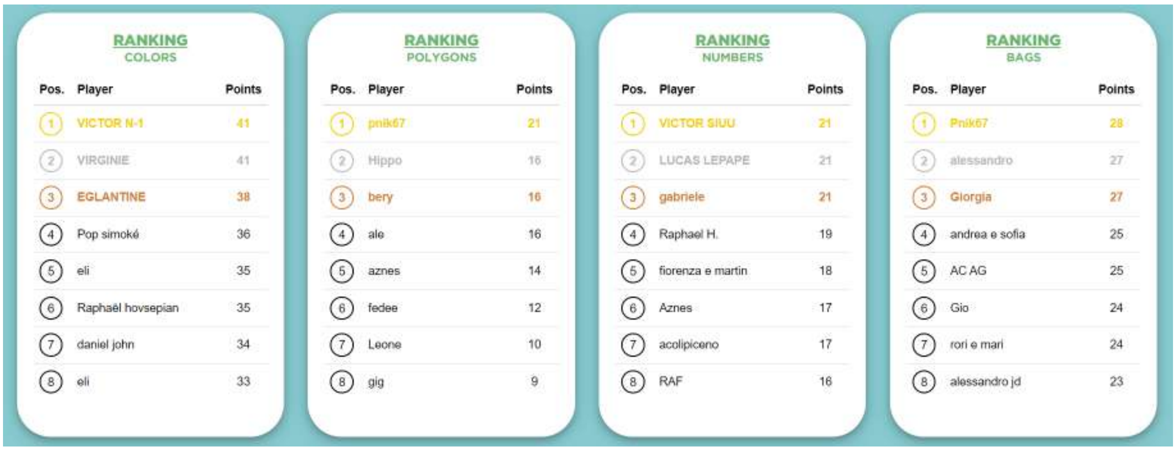


Figure 7.17.: There are four distinct leaderboards corresponding to four different environments.

take turns answering the questions, giving reasons for their answers. Considering what has been said about Game Semantics, students discussions were listened to and analysed, to try to pull out key points of the universal quantifier ∀ and its negation. Once the context of the game was clear, all students were able to answer the questions correctly. Although Level 1 of the colors environment eventually became easy for the class, this does not diminish its didactic value: firstly, a thorough understanding of the mechanisms in play is essential to move onto the more difficult levels with confidence and awareness; and secondly, working at a level at which students feel comfortable improves not only the accuracy of their answers but also their speed. This acceleration undoubtedly shows a deep understanding of the mechanisms used to generate these answers. The most interesting points come from the field of reasoning: not how a student answered but why. For example, when asked to discuss a turn like the one in figure 7.18, we had the three following conversations.

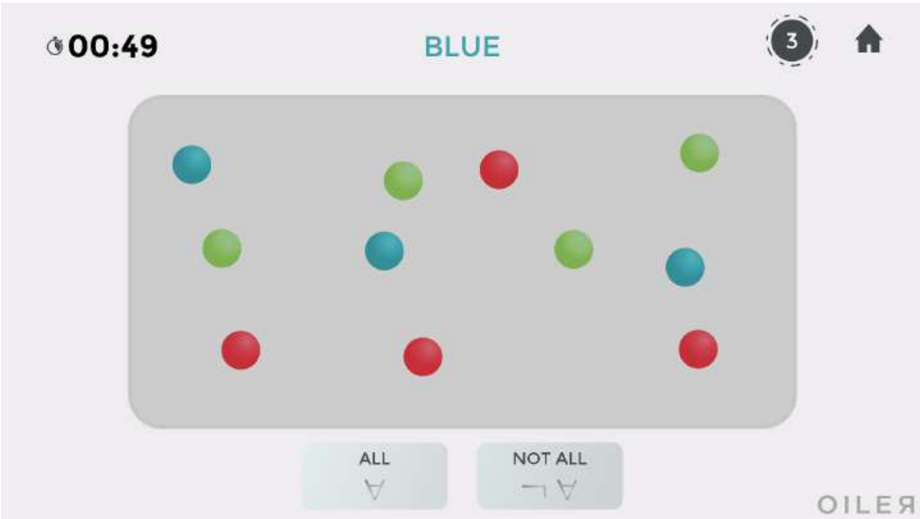


Figure 7.18.: Even though answering the question is not difficult, providing the right argumentation is essential to move onto the more difficult levels.

studentA:	Not all of them are blue.
teacher:	Why?
studentA:	Because *some are blue*!
studentB:	Not all of them are blue.
teacher:	Why?
studentB:	Because *only some are blue*!
studentC:	Not all of them are blue.
teacher:	Why?
studentC:	Because *some are green or red*!

In the first two exchanges the focus is on the blue balls (even though the second answer provides more precise information), whereas in the third exchange the focus is on the balls that are not blue, ignoring the blue balls entirely. This situation is of interest because it highlights the difficulty present from early years of education in proving the falsity of a universal quantifier—in other words, when trying to come up with a witness that does not have the property (a counterexample) as opposed to one that does (an example). As a similar difficulty is not encountered when discussing normal existential quantifiers, it seems that the cause of this struggle lies firmly in the negation, and perhaps in the scarce attention paid to this concept at school, as outlined in the literature. Indeed, if we limit ourselves to work with true statements or, more generally, with objects that satisfy a given properly, we lose the ability to "see": it is important to understand both *red* and *not red*; *triangle* and *not triangle*; or *greater than 3* and *not greater than 3* (which is not less than!).

Referring again to Figure 7.18, the fact that some balls are blue does not help us to reach the correct answer, as the answer would not have changed had there been many more blue balls or indeed no blue balls at all: not all balls are blue because at least one is either red or green.

The next step in reasoning with the universal quantifier consists in shifting focus from the set of red and green balls to one particular red (or green) ball: not all balls are blue because *this one is red*. As shown earlier, the "witness" setting in *Zermelo Game* asks players to do precisely this by clicking on a ball that does not satisfy the property. Independent play using this setting helps students to understand and accept the role of the witness. This understanding in turn improves their subsequent reasoning and helps to develop the meaning of the universal quantifier. Indeed, during sessions of independent play, it is common to catch students talking to themselves: verbalising the game procedures greatly aids their performance.

The existential quantifier ∃ is the result of improved comprehension in reasoning as well. When looking at the example in Figure 7.19, we find once again that that successful reasoning involves moving from "there are some green balls" to "this ball is green". This is likely due to the fact that the universal quantifier had been discussed previously, and that these two quantifiers are fundamentally connected.

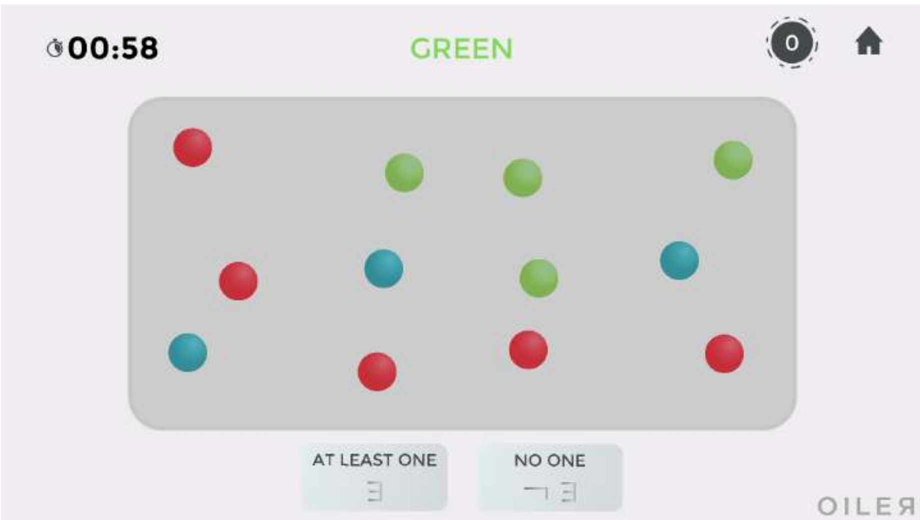


Figure 7.19.: The answer is that at least one ball is *green*.

The next exercise introduced the symbol ¬ to represent not. Students were explained that, within the context of the game, being ¬RED means to be either green or blue. Similarly, being ¬BLUE means to be either green or red, and being ¬GREEN means to be either red or blue.

The greatest difficulty the students faced was to identify an appropriate witness in a situation like that shown in Figure 7.20 and Figure 7.21.

An appropriate witness here is a red ball, because if not all balls are not red, then at least one is red. This way of using a quantifier between two negations is not only common in mathematics, but also in everyday language: if not everyone is untidy,

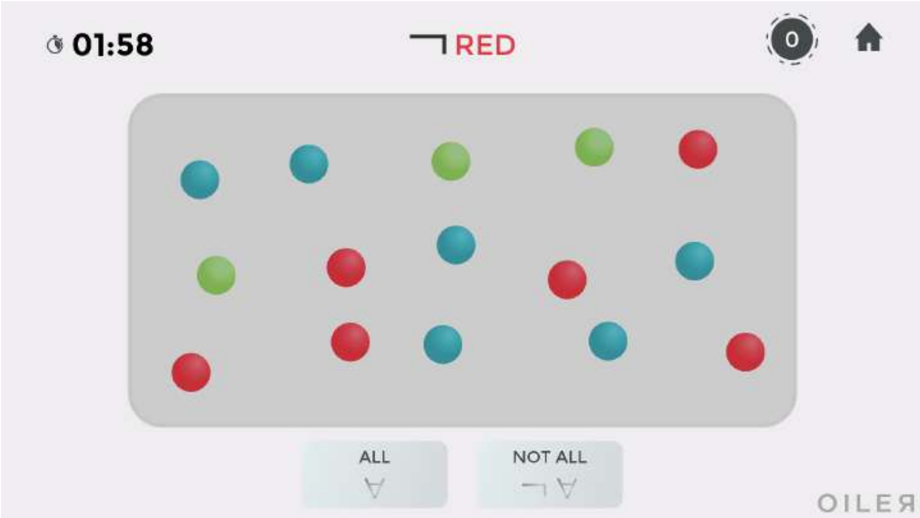


Figure 7.20.: The answer is that at least one ball is *not red*, because there is at least one green or blue ball.

then at least someone is tidy. If not all apples are bad, then there is at least one good apple, i.e., a non-bad apple. *Not being able to not go* means *being able to go*. The high school students faced similar difficulties, which we discuss in the following section.

7.4.2. High School

As we will discuss in more detail in the next Chapter 8, Zermelo educational path—along with the Bul—was adapted for some workshops aimed at high school students. In the next chapter, we will specify which activities were carried out and analyze the surveys that the students filled out at the end of the workshop. In this section, we will only analyze some dynamics observed during the Zermelo Game play phase.

Although the high school students played with all environments in *Zermelo Game*, we clearly feel it most important to discuss their experience with the bags environment.

Here we examine a discussion between a researcher and a student, where the student explains their reasoning to the researcher (Figure 7.22). Translating the formal notation, the question states "all the balls are blue", with a choice between "a bag exists" and "a bag does not exist". Upon choosing the answer ¬∃ BAG, the student defends their choice by saying "No, because here there is a red ball". Although the answer is correct, the reasoning is flawed, or at least incomplete.

With the "in this bag" being implicitly understood, that "here" uttered by the student implies that the bag used to defend the answer can be chosen by the player. Instead, the bag is chosen by the opponent (Figure 7.23). Therefore, in defending our answer, we need to explicitly state when the opponent puts our affirmation to the test by choosing a bag, we need to indicate a red ball, thus showing that not all balls within that bag are blue. Indeed, stating "there is no bag in which all balls are blue" is

Figure 7.21.: Primary school students playing on Zermelo Game with the *not red* propriety.

equivalent to stating "in every bag there is at least one red ball".

It is worth highlighting here that the last step of this reasoning, where we show that not all balls are blue by choosing a non-blue witness, is similar to that used in the colors environment. The bags environment is an environment that contains colors within it, but the statements made refer to sets of colors. The student's reasoning is therefore incomplete, or in some way implicit. A more comprehensive explanation would have been something along the lines of "If the computer chooses this bag, then the witness will be this ball [clicking on any red ball within that bag]".

Moving onto another exercise, the same problem comes up. "Is there a red ball in every bag?", the game asks. "Yes, there is it!", replies the student. In this case as well, the role of the opponent is implicit. At no point has the student mentioned the fact that the computer has made a move in choosing a bag.

When, instead, the two choices are made by the student (e.g., in the case of two existential quantifiers), then both the choice of bag and ball is explicitly mentioned by the student: "No, because in this bag, for example (first click), there is a blue ball (second click)." This is probably because the act of clicking on each element encourages the student to reason at each step. This aspect of game semantics is essential to understanding the concept of proof, and here we highlight the difficulty

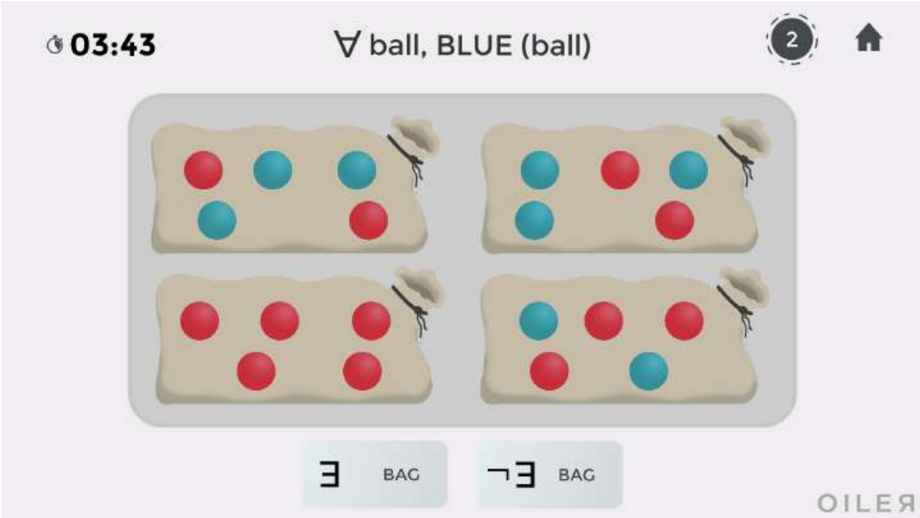


Figure 7.22.: The answer is ¬∃ BAG because no bag has all blue balls in it.

students may face in reasoning correctly. The game automates this process, and introduces it to the students. Indeed, after gameplay, the role of the computer in the bag environment was discussed and analysed.

Despite being fluent in mathematics, the student once again proposes a dynamic that appears to be shared by many others: having little familiarity with reasoning with quantifiers, and thus with quantifiers in general. Our hypothesis is that the use of *Zermelo Game* will help build this familiarity owing to the various dynamics discussed previously.

StudentD: There is a red ball in every bag, so I will choose a red ball.
Teacher: In a bag...
StudentD: ... that the computer has chosen.

In this case, the student has been pointed in the right direction and has concluded their reasoning correctly.

Zermelo Game was not designed to act as a stand-alone educational activity, but it becomes beneficial if integrated into an instructional path of introducing logic, as was proposed to primary school and high school students. Moreover, due to its arcade game nature, it can assist in preserving the learned concepts over time. *Zermelo Game* is not straightforward at its advanced levels, and both students and educators require time to devise the correct answer. However, the game triggers an almost spontaneous verbalisation, thereby keeping the connection between language and concepts alive and showing how the underlying meanings are indeed understood.

The report by A. Selden and J. Selden (2007) indeed discusses how one source of students' difficulties in discerning the logical structure of theorems is a lack of understanding of the meaning of quantifiers and that their order matters. Undergraduate students often consider the effect of an interchange of existential and universal quantifiers to have no effect. As we see, Zermelo Game is very useful for addressing this distinction.

Figure 7.23.: The environment chooses a bag and the player must show a red ball in it, proving that not all are blue in that particular bag.

8. Bul Educational Path

In *Bul*'s[1] educational path, the focus shifts from quantifiers to predicates and connectives, aiming to delve deeper into certain linguistic-mathematical aspects. The path involves a playful approach to logic using a range of tools and techniques: theatrical activity, simulation of so-called Boolean circuits, discussions about symbols, worksheets on predicates, solving equations by trial and error, and the online game. In this approach, the characters of the knave and the knight will also be present.

As part of the programme, we introduce formal symbols to identify certain logical concepts, such as predicates and negation. These are simple concepts, much like equality and addition, for which the necessity of symbolic representations from the first years of primary education is universally recognised. Just as the equality symbol supports the development of language and algorithmic thought, we consider the negation symbol to play an analogous role in the development of language and rational thought.

8.1. Background

Our programme focuses on the study of syntax and its relationship with semantics. Through their island, the knights and knaves can help to define and delimit the context of the analysis and work to be done: the symbols we introduce can only be used on their island—in other words, within a logical and symbolic context—and not, for example, in an essay. First, we focus on the syntactic aspects of language and how they come together to create meaning. To do this, we look both at the words that make up our language and the structure that supports it—i.e. the rules that allow us to move from words to syntactically correct sentences. As we will see, through symbolism, we can high-light the role of structure with respect to words. We argue that continually translating between the structure of a statement and its interpretation can favour "proceptual thinking" which, according to Gray and Tall (1994), is a key determinant of a "successful thinker" when it comes to development of cognitive and mathematical abilities. Indeed, the ambiguity of notation—*i.e.*, the role of a symbol both as a process and as a concept—"allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema"(Gray and Tall 1994, p. 115). Gray and Tall also believe that "mathematical symbolism is a major source of both success and distress in mathematics learning", and that a successful thinker is able to "employ the simple device of using the same notation to represent both a process and the product

[1]The name is a tribute to George Boole and Mary Everest Boole.

of that process". We believe that our activity on truth and falsehood and on sentence structure works towards the "ambiguity" (as per Gray and Tall) of meaning, even if the logical symbols we use do not correspond to the operational symbols used in Gray and Tall's examples. The importance of developing logically correct mental models at the primary-school level is highlighted in (Kuhn, Black, Keselman, et al. 2000). In this paper the authors argue that inquiry-based learning "at and from" middle-school level can be compromised by students having flawed mental models of causality. The paper shows how the simultaneous occurrence of a given value of a variable in a multivariable system and a particular outcome can be sufficient for students to expect a causal relationship between the variable and the outcome (in particular, the students struggle to conceive the outcome as being independent from the variable, and thus unaffected by the latter). The authors call this flawed model the *co-occurrence model*. This problem is not only due to a misunderstanding of causality, but also a failure to account for the additivity of the individual factors (i.e., their combined contribution) within a multivariate system. We believe that a true understanding of causality and additivity can only be reached after previous study of logical connectives. On the one hand, logical implication—which has no causal value—illustrates how co-occurrence is not sufficient for causality; on the other, the use of the connectives AND and OR with independent variables helps to develop the mental model required to "deconstruct" the total effect into that of the individual factors, providing the background required to understand additivity. Although the activity proposed here does not cover logical implication but in the online game Bul Game it discusses AND and OR connectives, as well as, in the first activities, it lays the necessary groundwork to address those topics.

To conclude this short section, it's worthwhile to briefly discuss Dependency Grammar. Dependency Grammar is a semantic approach to the structure of sentences, originally developed by French linguist Lucien Tesnière (1893-1954) and further advanced by scholars like Francesco Sabatini and Germano Proverbio in Italy. This framework differs from traditional syntactic analysis by focusing on the pivotal role of the verb. The valency of a verb, which is its ability to combine with essential sentence elements like the subject and complements, is described in terms of 'valency'. This concept of valency is analogously represented in mathematics by the *arity* of a predicate, which we will explore shortly.

8.2. Description of the path

The programme was carried out in various Italian primary schools and and a fourth-grade class of a French school. The description of the trial run is based on field notes and recordings from both primary schools, and some of the key educational moments of the programme are explored. The final quantitative analysis refers to two second-grade classes of an Italian school.

8.2.1. First Activity: Theatrical Activity

The goal of the first activity was to familiarize students with the characters of the knight and the knave. As this part overlaps with the content in Zermelo, further details can be found in the corresponding Chapter 7. The next part of the activity involves Smullyan's classic puzzles, where students had to work out whether the character speaking is a knight or a knave. This activity required two teachers, one who wore a knight or knave mask with their back turned to the class and provided riddles for the students, and another who helped the students to solve the riddles and identify which character was speaking. To start with, the riddles were very simple, as they did not follow the classic formulation seen in Smullyan's puzzles (which self-refer to the same group of characters speaking) but were simple statements such as "tigers can fly". The students were then asked in turn to play the role of the knight or knave and provide riddles for their classmates. The teacher then introduced the emblematic statement "I am a knight", always with their back turned to the class. After initial attempts to reach a decisive solution—during which both characters were suggested—the class realised that it was not possible to know whether the person speaking was a knight or a knave on the basis of that statement alone. Similarly, the class was encouraged to consider the phrase "I am a knave" and were pleased to discover that neither character would have been able to say this phrase.

At the end of the first part, some of Smullyan's simpler classic riddles were proposed to the class. These riddles involved more than one masked character, with their backs to the class, including one teacher and one or more students. The teacher instructed each of the masked students on what to say, while the rest of the class was tasked with deducing their identities. It's important to note that the activity was primarily aimed at the observing students, not those playing the roles of the knaves and knights, who were only required to repeat a given phrase. By rotating roles, this ensured that all students had the opportunity to engage with the activity as spectators. It is important to note that creating physical representations of the characters making these statements—with their backs turned and faces hidden, but nonetheless there in person—is likely to have made it easier for the students to solve the riddles. In the last part of the activity, Boolean circuits are introduced, using the knave as a representation of 'false' and the knight as a representation of 'true'. This choice works on a logical level, given that for every statement A made by a knave, we have $A \iff \perp$, and for every statement B made by a knight, we have $B \iff \top$. The aim in each circuit (see Figure 8.1) is to reach the final circle—shown in red in the figure—wearing a knight mask. At the start of the circuit (in the blue circle), the player chooses which mask to wear. The first circuit proposed is trivial, whereby the player simply has to follow the rope to the finish circle, with no unexpected events along the way.

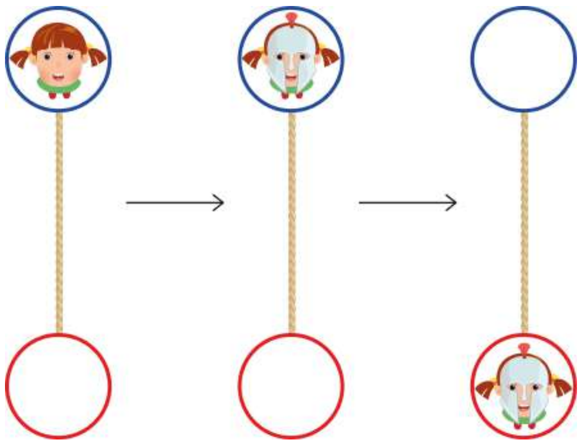


Figure 8.1.: A simple first circuit, whose solution is trivial.

The second circuit introduced Dr. No, a character (played by a student) who forces any player who encounters them to change their mask (Figure 8.2a). The winning strategy, as shown in Figure 8.2b, is to start the circuit wearing the knave mask.

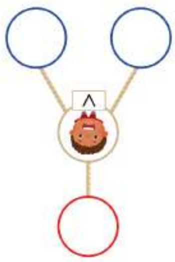
(a) Dr. No makes the mask change.

(b) To solve the negation circuit, you should start the circuit with a knave mask.

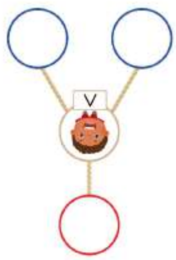
The next circuit then included two Dr. No's (Figure 8.3), one after the other; here, the winning strategy is to start the circuit wearing the knight mask. More Dr. No's were then introduced sequentially into the circuits, leading towards a discussion on the parity of the number of negations: if there is an even number of Dr. No's—including none at all—the winning strategy is to start with the knight mask; if there is an odd number of Dr. No's, the winning strategy is to start with the knave mask. After a few initial mistakes, all of the students understood the winning strategy and were able to choose the mask needed to successfully complete the circuit. The relationship between the winning strategy and the parity of the number of negations was highlighted.

Figure 8.3.: Two Dr. No's

The final circuits introduced the connectives AND and OR. Dr. AND (Figure 8.4a) is a character who **prefers knaves**: if approached by a knight and a knave, Dr. AND will let the knave pass; if approached by two knaves, they will let the knave of their choosing pass; and if approached by two knights, they will be forced to let a knight pass. Dr. OR is a similar but opposite character to Dr. AND, instead **preferring knights**: if approached by a knight and a knave, Dr. OR (Figure 8.4b) will let the knight pass; if approached by two knights, they will let the knight of their choosing pass; and if approached by two knaves, they will be forced to let one of the knaves pass.



(a) Dr. AND



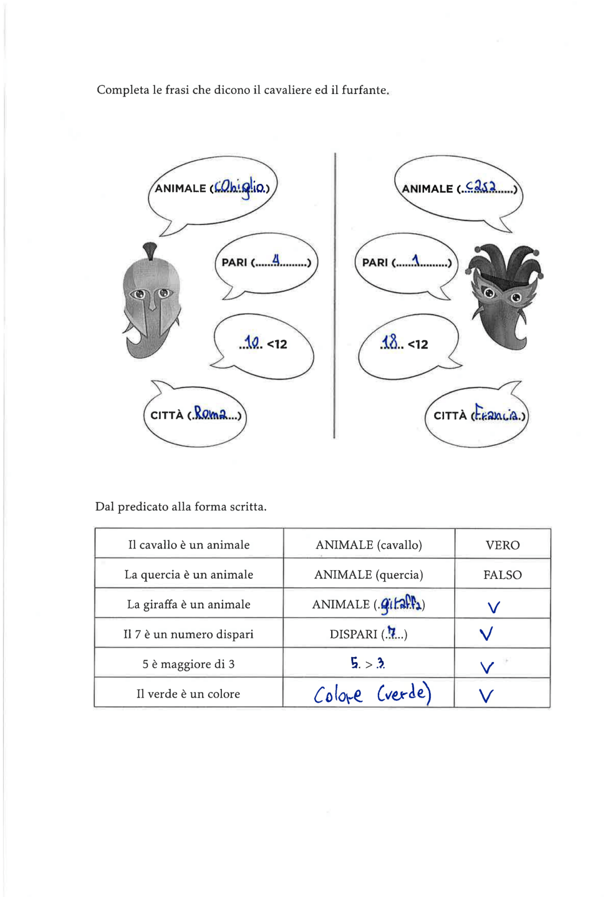
(b) Dr. OR

This way of using AND and OR corresponds exactly to the truth tables of the two connectives, positioning each connective as a rule of deduction rather than a symbol with a particular meaning.

Afterward, the students were given the freedom to design their own circuits to present to their peers. To achieve this, they selected a basic connective of the circuit (∧ or ∨) and then added negations where they deemed fit.

8.2.2. Second Activity: Predicates

The students were told that knights and knaves sometimes communicate with one another using a strange way of writing. First of all, students were asked to pick out the key elements of a phrase such as "a tiger is an animal", identifying 'tiger' and 'animal' as essential words to understand its meaning. More accurately, the central components of the phrase are the predicate "being an animal" and the object (in this case, the subject of the phrase) that the predicate refers to. The students were then told that knights and knaves use the two words 'tiger' and 'animal' and parentheses to write the phrase "a tiger is an animal". Some students suggested TIGER(ANIMAL) as a potential representation, and others (TIGER ANIMAL) (which is somewhat reminiscent of Barandrecht's lambda calculus!); a few other students suggested ANIMAL(TIGER). Each of these three notations can be used without leading to contradictions. The students were finally told that knights and knaves use the notation ANIMAL(TIGER). This is the standard notation used in logic and general mathematics, where the object of the predicate, or function, sits within parentheses after the symbol for the function. We feel that this early introduction of formal notation can be beneficial: first, it allows students to become accustomed to using a symbolic and context-dependent language (this language is used exclusively on Smullyan's island of the knights and knaves). This situation highlights the fact that changing language does not necessarily involve changing the vocabulary or alphabet; the formal language outlined here shares the same words and symbols as natural language (*e.g.*, English), but applies them using different rules. The key point is to create a broader view of language, which is not defined exclusively by its alphabet and vocabulary but also by the rules that govern the construction of phrases (Bernardi 2022). Furthermore, by considering a range of objects that either verify or falsify a given predicate, one is gradually able to identify and isolate the specific characteristics—*i.e.*, the properties—that characterise objects that satisfy that predicate. In other words, a notation such as ANIMAL( ) encourages the transition from an extensive description (based on many examples) to an intensive description of being an animal. Finally, the formal structure of predicates makes it easier to write phrases with the negation symbol, as we will see in the following phases. Importance was placed on the translation from symbolic form to natural language: TREE(OAK) should not be read "tree oak" but always "an oak is a tree". The students were given statements to translate in both directions, with examples of true statements—i.e., those made by a knight—such as ANIMAL(TIGER), and false statements—i.e., those made by a knave—such as ANIMAL(TABLE). To finish the activity, students were given exercises in which they were asked to correctly complete predicates according to which a character was speaking: for example, EVEN(...) or $3 < \ldots$ (Figure 8.5).



Completa le frasi che dicono il cavaliere ed il furfante.

ANIMALE (coniglio)

PARI (4)

10 <12

CITTÀ (Roma)

ANIMALE (casa)

PARI (1)

18 <12

CITTÀ (Francia)

Dal predicato alla forma scritta.

Il cavallo è un animale	ANIMALE (cavallo)	VERO
La quercia è un animale	ANIMALE (quercia)	FALSO
La giraffa è un animale	ANIMALE (giraffa)	V
Il 7 è un numero dispari	DISPARI (7)	V
5 è maggiore di 3	5 > 3	V
Il verde è un colore	Colore (verde)	V

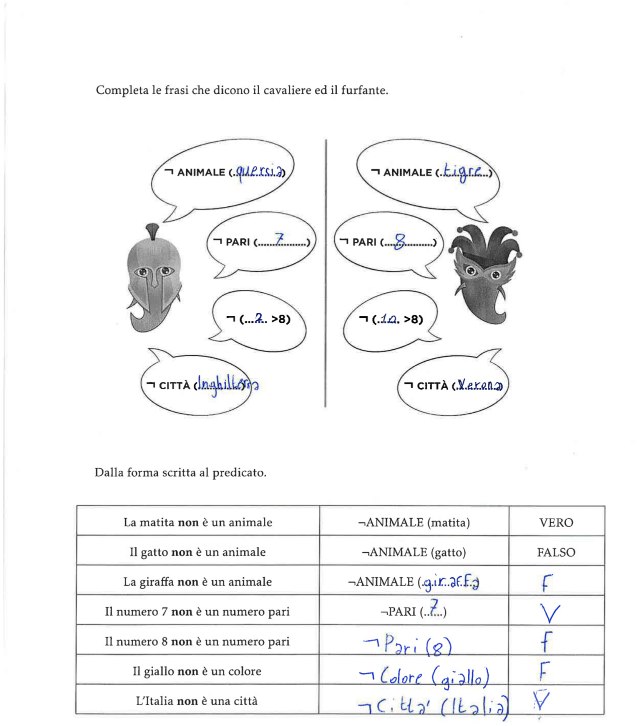
Figure 8.5.: An example of a student's work.

It should be noted that the students were free to fill in the predicates as they pleased. If a knight is speaking and we write CITY(x), x is necessarily a city. But if a knave is speaking, x can be anything that is not a city. Nonetheless, most students favoured the more meaningful contexts: CITY(FRANCE) makes more sense than CITY(9), even if both are wrong. Notably, with the predicates EVEN( ) and ODD( ), all students eventually opted for numerical contexts.

With older students, binary predicates involving two objects were introduced. The first binary predicate to be introduced was PARENT(x, y), with the teacher going through several examples with the students; the chosen convention was that x is a parent of y. Examples of this predicate were given where a knight was speaking, as well as where a knave was speaking. The predicate FRIENDS(x, y) was then introduced, with further examples. It was noted that writing PARENT(x, y) is different to writing PARENT(y, x) (in fact, one case precludes the other), whereas writing FRIENDS(x, y) is equivalent to writing FRIENDS(y, x). This latter property was described to the class as symmetry, which is common in mathematics: for example, the binary predicate $<$ is not symmetrical whereas the binary predicate $=$ is. Finally, the similarities between symmetry and commutativity were highlighted to the class.

8.2.3. Third Activity: Negation

The phrase “a tiger is not an animal” was written on the board and the students were asked, as in phase 2, to identify the key words. It was noted that, in addition to ‘tiger’ and ‘animal’, the word ‘not’ was also fundamental. A few exercises were done on the board whereby students needed to work out whether a given phrase had been said by a knight or a knave; for example, the first phrase was said by a knave, whereas the phrase “3 is not even” was said by a knight. The class was then given the negation symbol ¬ to colour in, to familiarise them with the symbol. Many recognised the symbol from the first phase, when it was used with the Dr. No character. Following this, the students carried out translation exercises—first orally at the board, and then written—and were given comics to fill in, depending on whether the person speaking in the comic was a knight or a knave. The phrase “red is not a colour” would be translated as ¬ COLOUR(RED). Similarly, the phrase ¬ ODD(4) is translated as “4 is not an odd number”. We highlight here that two different approaches were taken for the negation symbol. In phase 1, the symbol was introduced as a rule: the symbol acted on the truth value of a statement by changing it — that is, by changing the mask worn. In phase 3, the negation symbol was introduced as a logical connective with semantic value. These two interpretations are clearly very closely connected. If either a knight or a knave writes the phrase PREDICATE(OBJECT), then the introduction of the negation symbol will force a character swap, because the phrase ¬PREDICATE(OBJECT) can only be written by the other character. Until this point, the statements provided that contained the negation symbol had been limited to the form ¬PREDICATE(OBJECT). The statement $\neg\ (3 < 2)$ was then written on the board and a student was asked to translate it. Surprisingly, the student translated it as “3 is not less than 2”, applying the negation to the predicate. Exercises have been proposed to review the negation (Figure 8.6).



Completa le frasi che dicono il cavaliere ed il furfante.

Dalla forma scritta al predicato.

La matita **non** è un animale	¬ANIMALE (matita)	VERO
Il gatto **non** è un animale	¬ANIMALE (gatto)	FALSO
La giraffa **non** è un animale	¬ANIMALE (giraffa)	F
Il numero 7 **non** è un numero pari	¬PARI (7)	V
Il numero 8 **non** è un numero pari	¬Pari (8)	F
Il giallo **non** è un colore	¬Colore (giallo)	F
L'Italia **non** è una città	¬Citta' (Italia)	V

Figure 8.6.: An example of a student's work in an exercise involving negation.

8.2.4. Fourth Activity: Variables

This part of the programme concerns the search for a solution via trials and errors. We note that a single variable equation is a particular type of unary predicate. Use of the knave character seems to help children through the analysis stage by removing the fear of making mistakes. The main exercise in this activity involved laying out on the floor many cards showing numbers and formulas containing an x. The aim was to complete the equations, inequalities, or predicates—such as EVEN(x)—by placing an appropriate number over the x; in other words, substituting a constant for a variable. To make students comfortable with the notation, the variable x was firstly introduced as a mystery number. Questions such as "I know a number x such that $x + 3 = 8$. What number is it?" or "I know a number x which, when added to itself, makes 10. What number is it?" were posed. It was not hard for the students to answer these first simple questions. Such equations, which are generally first encountered in middle school (ages 11–14 years), are usually solved using a *synthesis*[2] method that relies on inverse operations. We clearly did not consider it appropriate to introduce such a method

[2]Opposed to *analysis*.

at primary school and the equations were instead solved by trial and error: different numbers were substituted for x and the resulting equality was checked. More complex statements were proposed to the class, such as EVEN(x), and the class noted that, this time, there were many different possible solutions. Multiple requirements were therefore added together: "I know a number x such that EVEN(x), $x < 10$, and x is a three-letter word. What number is it?" In this case, there is still more than one solution, but the number of solutions is finite. The children were allowed to work freely, and enjoyed coming up with numbers they wanted to substitute for x, trying out a wide range of numbers. By that time, the class was used to recognising false statements (and judging them as such) thanks to their familiarity with the knave character. We believe that solving through trial and error should also be encouraged in older year groups to make students more comfortable with errors and falsehood. Once an equation was solved by a student, it was put aside (Figure 8.7a and 8.7b).

(a) (b)

Figure 8.7.: The students solve the equations by trial and error using the numbers arranged on the floor.

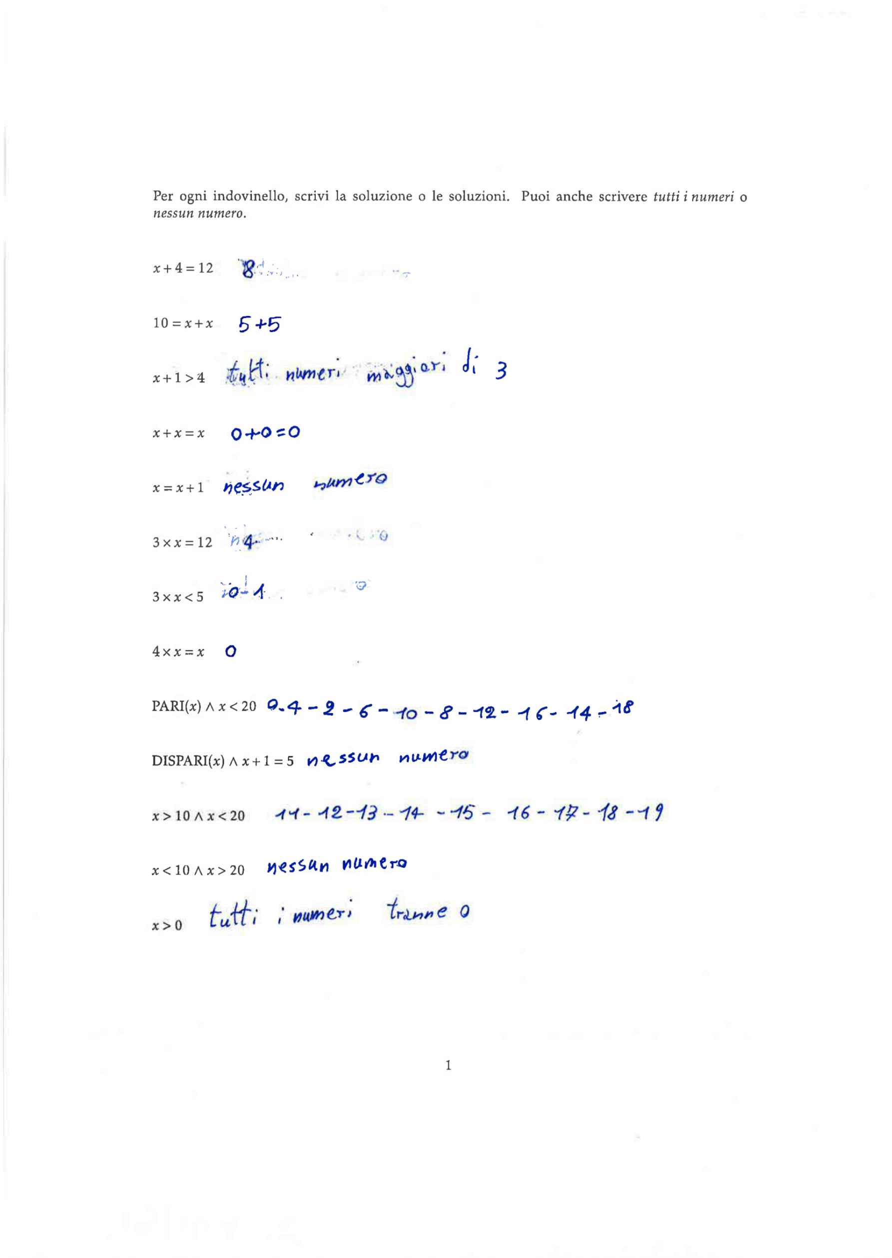
This trial-and-error approach is an excellent example of roaming freely through mathematics. The activity described above offers the perfect opportunity to discuss proceptual thinking, as outlined in theoretical framework. Children of primary-school age are not yet able to manipulate equations (*i.e.*, moving elements from one side to the other) and thus cannot carry out the "process" represented by the equation: an equation—before any substitutions—is simply a concept. Once a constant (number) has been substituted into the equation, the equation is processed. If the equality obtained is false, you go back to the original concept, more knowledgeable than before.

Bul path is clearly linked to Zermelo's one, and if the class has already tackled it—or part of it—, one can reflect on quantifiers in relation to the activity.

For example, a formula was written on the board that contained x and the students were asked whether the elements x that satisfied the formula (referring implicitly to natural numbers) were all, some (*i.e.*, at least one; the students did not seem to have a problem with it being exactly one), or none. For example, $x+3=5$ is satisfied by one number, whereas $x=x$ is satisfied by all numbers; by contrast, $x>x$ and $x+3=1$ are not satisfied by any natural number. The same question was posed about the predicates EVEN(x) and ODD(x), noticing that, even if not all numbers satisfied the predicates, both were satisfied by infinitely many numbers. Furthermore, it was pointed out that EVEN$(x+x)$ is satisfied by all natural numbers. The class was asked to find an equivalent expression such that ODD(expression) was true for all natural numbers. At first, the class had no idea how to approach this problem, but then began to work out what sort of expression would be required. They were placed into small groups to work on a solution, supported by three teachers. The students suggested solutions such as ODD$(x-x+1)$: they were told that, while correct, these expressions always give the same result, regardless of the value of x, and were encouraged to find a non-constant expression. After a while, several students independently concluded that a possible solution was ODD$(x+x+1)$.

8.2.5. Fifth Activity: Conjunctions and Disjunctions

Once predicates and negation have been discussed, the educational path can be concluded by introducing the main connectives *and* and *or*. Remember that the connectives have already been dealt with at the level of logical rules in the circuits phase. Before explicitly discussing their meaning and role with the class, another activity that is particularly interesting is—picking up on the equations proposed in the previous phase—to add the requirement to use the $\wedge$ connective in solving the equations. In other words, two equations are chosen and joined with the $\wedge$ connective: the student's task is to find an x that satisfies both predicates, if it exists. Once this first phase is completed, since the class had already gone through some stages of Zermelo's path, the following exercise was attempted, shown in Figure 8.8, where for each equation it was necessary to indicate which natural numbers satisfied it.



Per ogni indovinello, scrivi la soluzione o le soluzioni. Puoi anche scrivere *tutti i numeri* o *nessun numero*.

$x + 4 = 12$ 8

$10 = x + x$ 5+5

$x + 1 > 4$ tutti numeri maggiori di 3

$x + x = x$ 0+0=0

$x = x + 1$ nessun numero

$3 \times x = 12$ 4

$3 \times x < 5$ 0-1

$4 \times x = x$ 0

PARI$(x) \wedge x < 20$ 0-4-2-6-10-8-12-16-14-18

DISPARI$(x) \wedge x + 1 = 5$ nessun numero

$x > 10 \wedge x < 20$ 11-12-13-14-15-16-17-18-19

$x < 10 \wedge x > 20$ nessun numero

$x > 0$ tutti i numeri tranne 0

1

Figure 8.8.: For each formula, we asked to indicate which natural numbers satisfied it.

At this point, the logical and linguistic value of the connectives can be discussed with the class, also with the aid the Bul Game, which we will see in the next section. The Dialogue Rules mentioned in Chapter 6 will also be discussed.

8.2.6. High School Workshops

In addition to experiences with primary schools, four two-day workshops were conducted with high schools. The workshops aimed to introduce logical language and argumentation, following the same structure as the Zermelo and Bul educational path, albeit at a different pace. In one of the four workshops, part of the Lovleis pathway was also introduced, which will be the subject of the next chapter 9.

In the course, predicates and connectives were introduced to the class, offering insights on how connectives are present in various everyday situations. At this point, the class was allowed to pair up and play Bul Game (selecting all connectives) to confirm their understanding of connectives and their corresponding truth tables. While maintaining a relaxed atmosphere during the Bul Game session, competition was encouraged by displaying the leaderboards online with the top scores so that all players could see them. Attention then shifted to quantifiers, and subsequently, Zermelo Game was played, an activity discussed in the previous Chapter 7.

Subsequently, an activity was proposed that particularly intrigued the students. The class was divided into groups, and each group received a sheet containing only some symbols. The goal for each group was to understand the relationship between the shown symbols. More precisely, the five sheets displayed in Figure 8.9 were distributed.

The first two images aim to highlight the relationships on one side between $\wedge$ and $\forall$, and on the other between $\vee$ and $\exists$. The two images in the second row refer to the Aristotelian square, both in its classical form and with the connectives (*i.e.*, De Morgan's laws). The last image addresses the problem of the relationships between the two quantifiers. The various groups then presented their findings to the rest of the class, trying to reach a shared conclusion. In particular, the groups were usually able—with a little help—to identify both the similarities and differences between $\wedge$ and $\forall$ (as well as between $\vee$ and $\exists$) and the relationships both in the Aristotelian square and between the quantifiers. However, more difficulty was encountered in identifying and understanding De Morgan's laws, which required additional discussion.

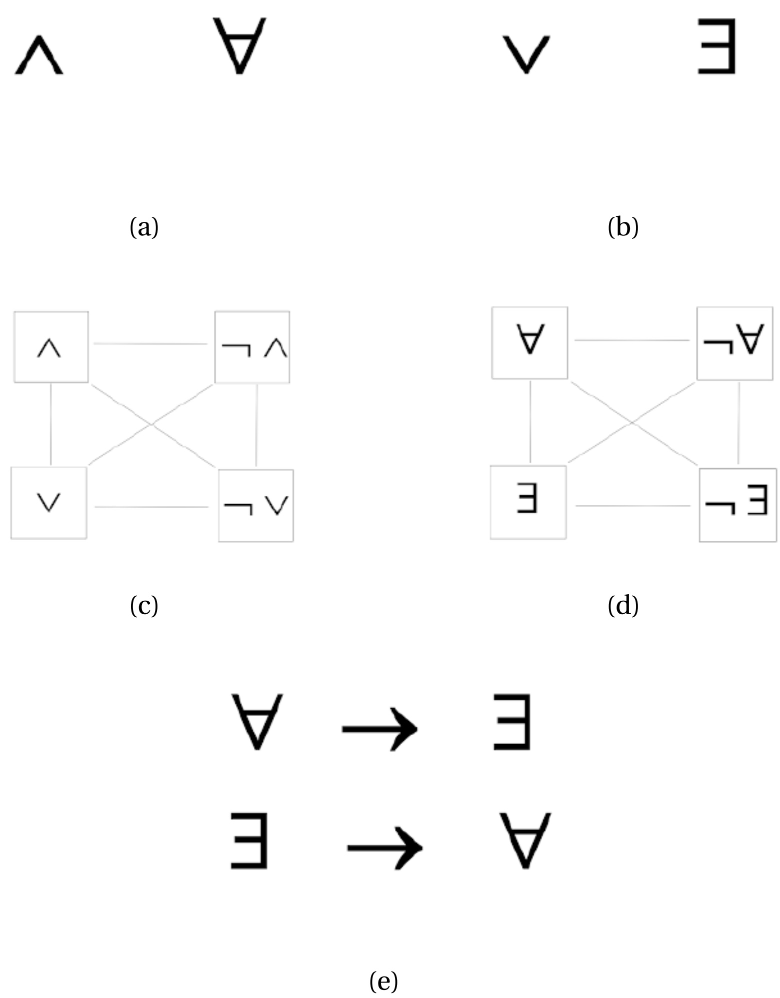
Depending on the stage, the course took different directions: for example, in one group, the principle of induction was discussed; in another, the path of Lovelace was explored.

8.3. Bul Game

The software *Bul Game*, accessible at `www.oiler.education/bul`, is a free online game designed to support didactic activities about logic and its connections with everyday language[3]. The aim of the game is to make correct choices based on statements made by knights, who always tell the truth, and by knaves, who always lie. The statements may involve predicates, connectives, negation, and implication (Bernardi 2022).

The game was designed by myself and developed by Jacopo Zuliani, Mattia Sanchioni, Giulia Balboni, and Martina Carbone employing classic online game elements to enhance student motivation and engagement in learning logic through captivating visuals, adjustable difficulty levels, self-competition and rankings, self-paced learning, and instant feedback upon errors.

[3]The game is available in English, French, and Italian.



(a) (b) (c) (d) (e)

Figure 8.9.: Worksheets proposed to the various groups, one per group.

8.3.1. The Language

The notation used to describe predicates is typical of logic, but it also recalls the ordinary notation used for functions, *i.e.*, $f(x)$. In fact, each predicate is a function that associates to one or more objects the values 0 or 1 (*i.e.*, false or true). Another symbol that is used in the game is the negation symbol ¬, which must be understood as *not*. For our purposes, the not is only placed in front of a predicate—and not in front of a more complex formula—to obtain a notation of the type ¬ANIMAL(tiger), which should be read as "a tiger is not an animal". In the game, we find predicates referring to general knowledge, e.g. TREE(oak), as well as predicates referring to mathematical context, *e.g.*, EVEN(3). We underline that statements such as $3 < 2$ are binary predicates, where the uncommon notation $<(3, 2)$ is avoided. The classical logical connectives of conjunction ∧ and disjunction ∨ are also used in the game, as well as the symbols of implication → and exclusion which will be discussed further on.

8.3.1.1. How to Play?

The game is played by pressing the A and B buttons—using the keyboard or on the screen—according to the clues given by knights and knaves. If you press the correct button, you score 1 point; if you make a mistake, the game ends. The aim of the game is to score as many points as possible in a given time. In the homepage menu you can choose how to set up the game choosing the type of questions given (true&false, predicates, predicates with negation, . . .), the type of predicates which will appear in the clues—general knowledge or mathematics—and the time available for your play (Figure 8.10).

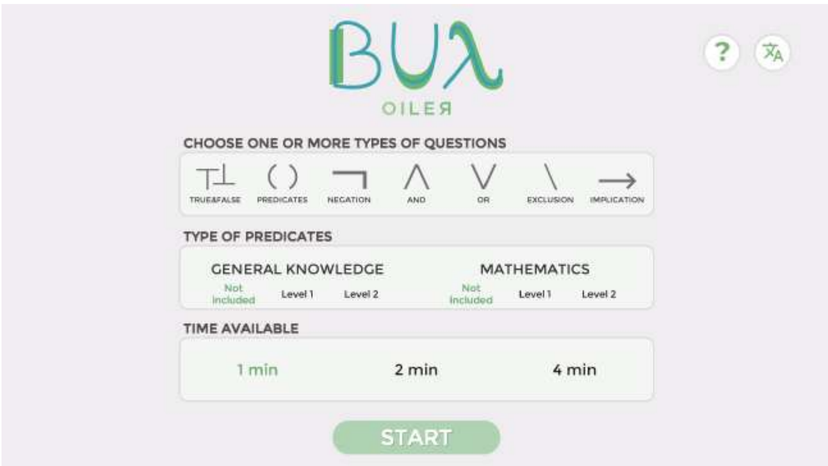


Figure 8.10.: In the homepage menu you can choose how to set up the game.

All questions that appear in the game are randomly generated for each game, so it may happen that the same question appears twice in a row. Let us now have a closer look at the type of questions in the game. In the first kind of exercise, it is simply a matter of recognising true and false statements—-that is, paying attention to the character making them. At each turn, you need to press either A or B to continue (if you do not press anything, you will not move on to the next turn), see for example Figure 8.11. In particular, we will not follow the knave's clues but will instead do the opposite of what they suggest: the most complicated case is when a double negation appears (from which we can see that we are in a classical logic-type environment). If the knave says "the path continues by not pressing B", we will have to press B regardless.

Figure 8.11.: The right answer is to press A.

In the predicates section, phrases such as ANIMAL(tiger) could appear together with mathematical predicates. For instance, in Figure 8.12, we see a knight with the speech bubble "If $24 + 7 = 31$ the path continues by pressing B, otherwise by pressing A". Since the computation is correct, and the knight says the truth, we need to press B to continue.

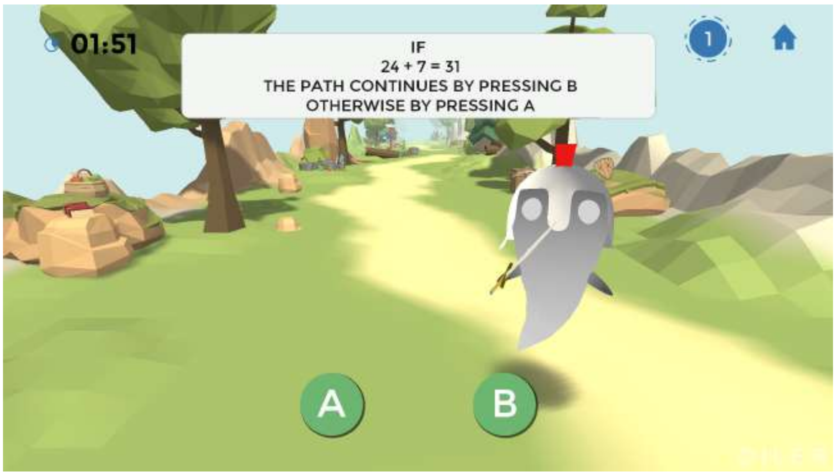


Figure 8.12.: The right answer is to press B.

The negation section introduces the not symbol (*i.e.*, ¬). ¬COLOR(apple) should be read, as discussed, as "an apple is *not* a color". For instance, in Figure 8.13 we see a knight with the speech bubble "If ¬ ANIMAL (Paris) the path continues by pressing A, otherwise by pressing B". Since Paris is not an animal, and the knight says the truth, we need to press A to continue.

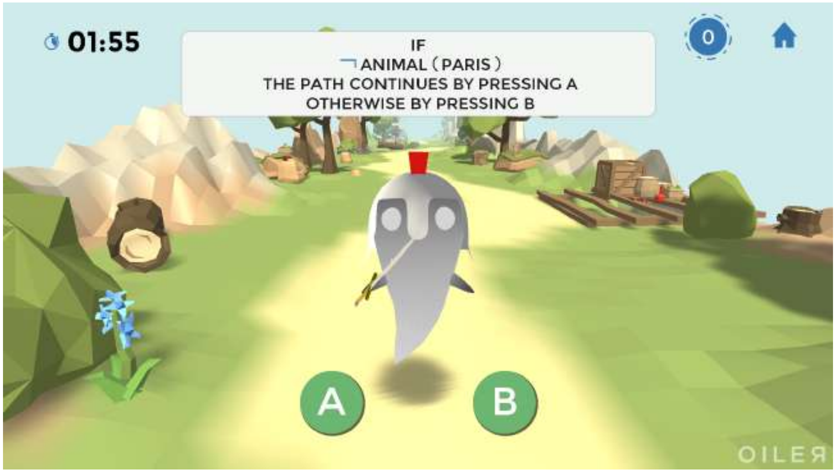


Figure 8.13.: The right answer is to press B.

The $\wedge$ and $\vee$ symbols are then introduced to obtain formulas of the type TREE(oak) $\wedge 2+2=4$. For instance, in Figure 8.14 we see a knave with the speech bubble "If $62 > 45 \wedge 76 > 88$ the path continues by pressing A otherwise by pressing B". Since $76 > 88$ is false, the conjunction is false.

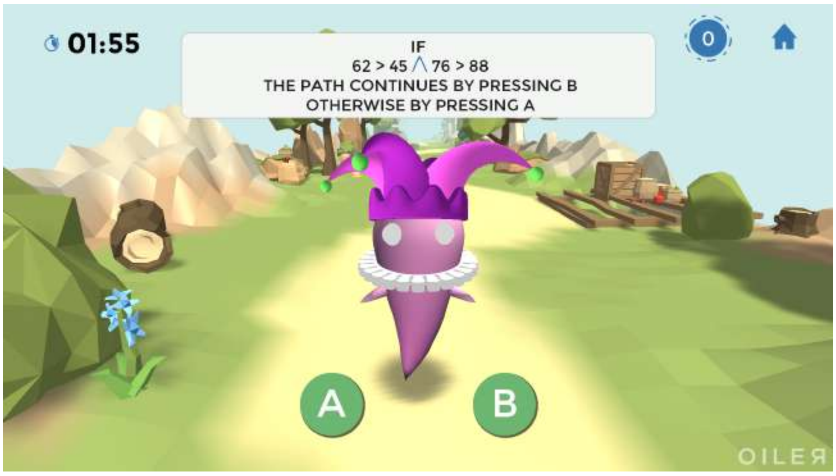


Figure 8.14.: The right answer is to press B.

Instead, in Figure 8.15 we see a knight with the speech bubble "If CITY(mouse) $\vee$ ANIMAL(eagle) the path continues by pressing B otherwise by pressing A". Since an eagle is indeed an animal, the right answer is B.

The exclusion connective $P \setminus Q$ is true if and only if P is true but Q is false. For instance, in Figure 8.16, we see a knight with the speech bubble "If ODD(5) \ EVEN(11)

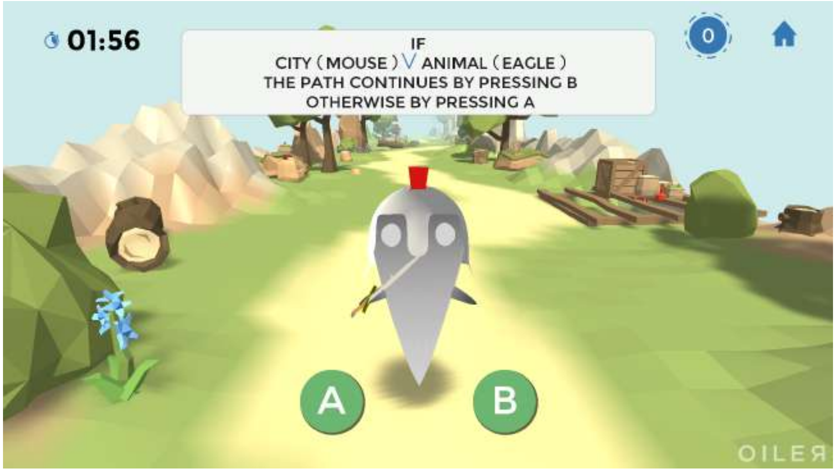


Figure 8.15.: The right answer is to press B.

the path continues by pressing B otherwise by pressing A". Since 11 is not even, the right answer is A. The last type of question is about the implication $P \rightarrow Q$ (*i.e.*, the negation of the exclusion). In particular $P \rightarrow Q$ is false if and only if P is true and Q is false. For instance, in Figure 8.17 we see a knave with the speech bubble "If MONTH(October) → MONTH(London) the path continues by pressing B otherwise by pressing A". Since false implies true is false, the right answer is to press B.

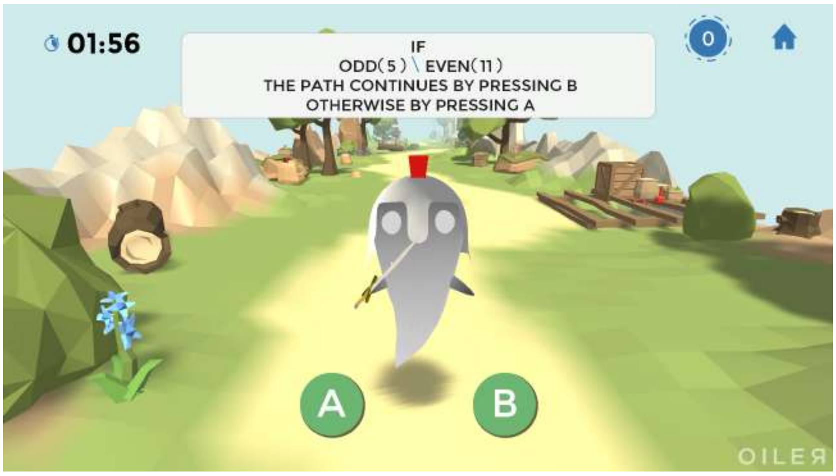


Figure 8.16.: The right answer is to press A.

8.3.2. Is the Game Logically Consistent?

The content of the present subsection is beyond the scope of school curricula and wants only to show that the game is consistent. Let us start with a clarification: when, for example, the knight says "The path continues by not pressing A", a priori also not pressing any key is a correct action. In the game, there is a tacit agreement that it is

Figure 8.17.: The right answer is to press B.

mandatory to press either A or B (*i.e.*, A aut B) in every turn. Now, consider a statement such as "if X then Y else Z". The sentence should be interpreted as follows: if the condition X is fulfilled, then do Y, otherwise do Z. The clues such as "if TREE(oak) then the path continues by pressing A otherwise by pressing B" that appear in Bul Game are exactly of this type. Because of what we have previously said, pressing B (or A) is the negation of pressing A (or, respectively, B). So, in our case, in the statement "if X then Y else Z", Z is always of type ¬Y. This makes saying "if X then Y else Z" equivalent to saying "if X then Y else not Y" (i.e. "X if and only if Y", which is $X \iff Y$). Now let K be a predicate for characters (*i.e.*, K is true if and only if a knight is speaking) and let P be any statement. Suppose a knight is speaking and P is their statement: it will be verified that $K \iff P$, since both are true. But even in the case of a knave speaking, $K \iff P$ is verified, since both K and P are false. In general, if a character asserts P, it is always verified that $K \iff P$. In our game, in particular, we have said that any utterance P made by knaves or knights is of the type $P = X \iff Y$. We then have that every turn in the game can be rewritten as $K \iff (X \iff Y)$ (the speaking character appears explicitly, so K is always known). So, if K is not verified (*i.e.*, the speaker is a knave) then $X \iff Y$ must also be false (*i.e.*, the truth values of X and Y must be different). For example, if the sentence spoken by the knave is "if COLOR(blue) press A", since COLOR(blue) is true, "press A" must be false, which means that pressing B is the right answer. By contrast, if the speaker is a knight, then K is verified and thus the truth values of X and Y must coincide.

8.4. Analysis of High School Questionnaire

As mentioned in the description, the educational path of Bul—alongside the paths of Zermelo (Chapter 7) and Lovleis (Chapter 9)—was adapted for several workshops aimed at high school students. The focus was on connectives, quantifiers, and logical language in general, not just in its relationship with mathematics but also with everyday language. Specifically, four workshops were conducted, involving a total of 88 students, of which 26 were French and the rest Italian. The classes were often mixed, with students of different ages and from different schools.

In this section, we aim to analyze the questionnaires that students completed anonymously at the end of the experience. A questionnaire was administered to 22 students asking what they liked most, what they liked least, and what they felt they had learned. For the remaining students, in addition to questions about what they liked most and least, they were explicitly asked for their opinion on symbolism. Overall, the evaluation of the educational path was positive in every questionnaire. Furthermore, 63 out of 66 students acknowledged the fundamental role of symbols in their responses about symbolism. Below, we present the responses we consider most relevant from the questionnaires, attempting to highlight the most common types of answers.

Let us start by analyzing some responses from those 22 students who were not explicitly asked about symbolism. Nonetheless, symbolism played a central role in many answers. The responses presented here refer to the question *what do you feel you have learned?*.

> I feel I have come to better understand the symbols, the relationship between the signs because they always seemed useless to me, and very often during class assignments, I would not use them.

> I feel I have learned to use logic well and the true meaning of 'for all', 'exists', 'not for all', 'does not exist' along with their relationships, similarities, and differences.

> I feel I have learned the meaning of some mathematical symbols that I had not fully understood.

> I have come to understand much better things that I had never grasped in three years of high school, and now, unlike before, I find them interesting.

This type of response highlights a specific dynamic at the school level: on one hand, there is a lack of proper introduction to symbolism; on the other, symbols are commonly used. Thus, in teaching practice, an implicit contract is established where symbolism is to be used even if the student does not recognize any key or facilitating role to the symbols.

> The part I liked the most was understanding the relationships between the symbols and explaining it to the class, because it was mentally stimulating.

> I understood the connectives well, having a lot of fun, and not feeling any boredom or sense of heaviness.

> Among the things I particularly appreciated the most was the fact of having analyzed the concept of a winning strategy in games in relation to logical language. It was also very interesting to have the opportunity to independently or in a group demonstrate the deep connections existing between different connectives or quantifiers (as in the case of $\forall$ and $\wedge$, which in my opinion is the most fascinating case)."

Many responses, like the ones mentioned, view the study of symbols as something mentally stimulating and fun. It's important to note that symbolism is brought up without any explicit request regarding it.

> To think more and faster.

> I feel I have learned to reason better and to express a mathematical rule with a formula.

> The ability to reason and the speed in doing so.

> I have learned to use *logic much more than theory*.

In this type of answers, it's interesting to note how logic emerges as related to reasoning, to motivate and explain the processes being followed. In this direction, the proposed opposition between logic and theory is intriguing: somehow, logic is seen as the tool to delve into the meaning of things and not to trust on theoretical grounds. A shorthand component of logical language is also emphasized.

Let us now move on to analyze the other questionnaires that included an explicit question about symbolism, thus focusing on the responses to this.

A common type of response, as can be seen from those mentioned, emphasizes the connection with language, clarifying how symbols prove useful in reasoning not only in mathematics but also in understanding everyday language better and avoiding ambiguity.

> It's as if symbols were *another language* or rather another way of speaking. I believe that initially, it may be difficult to get the hang of it, but with some practice, *symbols transform into concepts*; moreover, I think that *using symbols opens up more possibilities for dialogue*, considering that many things we say can be described through the use of symbols.

Interesting and practical topics, *present in life*.

Until now, I did not understand their use very much, but now I am aware that they are also commonly used and can be helpful in everyday life.

These are topics that are seldom covered at the school level, but they can prove useful in various situations.

I believe they are really interesting and *beautiful to understand*. I've realized how, thanks to them, *it is always possible to simplify and make things understandable to everyone*, and especially how they are closely linked to everyday life.

I think that at first glance, it may seem like a boring and perhaps even complicated topic, but I believe that if it is explained with the right approach, it can become a fascinating subject that *opens the eyes*.

I discovered that symbols are present in the sentences we utter daily, and therefore, in part, they are not difficult to understand. I find that their use in mathematics is aimed at simplifying complex sentences or conditions.

I think they're very useful, both for mathematics and beyond. In fact, during these two days, it was very interesting to learn about the *close link between mathematics and real life through these symbols*.

Useful for training the brain and *understanding Italian*.

I had never thought that in our daily lives we often use symbols without realizing it, and this also *makes me more aware of what we say and how our language works*.

Not knowing these symbols before taking this course, I had no idea how important and present they are in our daily lives, especially for simplifying and synthesizing sometimes very complicated concepts.

> The use of symbols is certainly very useful, they promote *direct communication free of possible contradictions and misunderstandings*, which is essential in mathematics.

> In general, I believe that symbols, even just the =, were created to simplify. They are therefore useful, perhaps in some contexts more than in others. This particular one more than others.

> It's a type of language not common to us that, however, turns out to be fun to use also for better understanding *meanings and negations of sentences in Italian.*

Another interesting type of response recognized symbols as having a role of universality, independent of the spoken language.

> I find that the inventor of these symbols was very intelligent because they allow expressing logic without going through French.

> I find that the use of these symbols is very important because it's a *universal language that everyone understands.*

As already mentioned, not all responses were positive, with 3 out of 66 not recognizing a particularly interesting role for symbols. Specifically, the three answers where the follows.

> I understood them well, but I wonder what use they might have in "non-mathematics".

> I find they are not very useful if one does not do mathematical logic. It's possible they are very useful.

> Sometimes they are useful, but surely I will not use them in daily life, I think they are a succinct way to summarize everything and useful in some cases.

We note that the third student, despite the relatively negative response, makes use of them—in an ironic yet correct fashion—in answering the request to provide suggestions for improving the workshop: "$\neg\exists$suggestion", the student reports. Regarding the other two responses, we note that they were given in the French class. Maybe, being an activity so closely linked to language, my not mastering the language well also had a significant impact.

In recalling that all surveys were completed in a fully anonymous form, let us now gather the most recurrent opinions. Generally, the words "interactive" and "dialogical" frequently emerge in the surveys, along with an appreciation for group work, which evidently some students are not accustomed to. Regarding symbolism, firstly, the universality of symbols was noted, with the term "universal" appearing 5 times within the surveys. Generally, symbols are seen as something closely linked to language and reasoning, useful both for their conciseness and for their lack of ambiguity. Many students also highlighted how, despite not being adequately addressed in school, symbols are nonetheless utilized. Our hypothesis, supported by these surveys, is that symbols are not necessarily perceived as irrelevant abstractions, but if they are, it is likely due to a lack of proper introduction. We hope, as is already the case in some high schools, for a serious discussion on symbols and their connection to language, as well as on what were referred to as *Dialogue Rules* in Chapter 6. Furthermore, as has been extensively discussed, we hope that symbols can be introduced before students reach high school.

8.5. Quantitative Analysis

In this section we explain how we assessed the causal impact of the Bul educational programme in the development of cognitive skills and mathematical literacy in the primary school, in order to provide empirical evidence supporting our main argument. The analysis is structured as a randomised controlled trial (RCT). This analysis was conducted together with Riccardo Manghi, a research fellow in Economics at LUISS Guido Carli in Rome.

8.5.1. Methodology

The empirical exercise was structured as follows: we selected two different second-grade classes in the Italian primary school. One class was the intervention group, in which the educational programme Bul was run, and the other class served as the control group, receiving the usual programme of teaching. To avoid potential sources of endogeneity, the control class was in the same year as the intervention class, and shared the same teachers. We validated the RCT assumption of randomisation by using the proper balancing test to check that class compositions were as good as random with respect to relevant covariates (age, sex, and nationality of origin), and then measured mathematical literacy and cognitive skill in both classes, before and after the intervention, and calculated the score difference. We measured mathematical literacy using INVALSI questions. INVALSI are national tests specially designated and recognised by the Italian state to evaluate skills in fundamental areas such as mathematics, Italian, and English. Questions from the mathematics INVALSI tests therefore provide a good measure of mathematical literacy. The pre-intervention and post-intervention tests used in this trial were composed of different sets of four past INVALSI questions. An example of one of the INVALSI questions used is shown below (Figure 8.18):

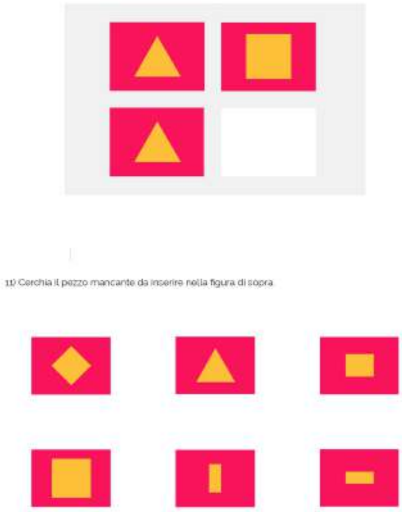


Figure 8.19.: The student is asked to choose the missing piece.

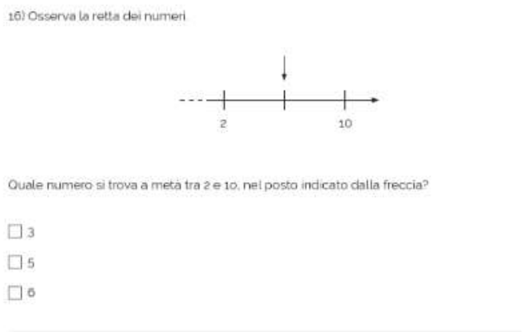


Figure 8.18.: In the question, the student is asked to say which number lies halfway between 2 and 10.

We measured cognitive skills with Raven's progressive matrices. This non-verbal test is widely recognised as a measure of fluid intelligence, which refers to the ability to solve novel reasoning problems, and is correlated with several important skills such as comprehension, problem solving, and learning in individuals aged 5 years and older (Kaplan and Saccuzzo 2009). We used 13 questions from the Colored Progressive Matrices (RCPM) variant, a version of the Raven test designed specifically for children aged 5–11 years (G. Domino and M. L. Domino 2006). An example question is shown below (Figure 8.19):

We considered this type of test to be a reliable proxy of general cognitive skills, as a review of the psychological literature suggested that no factor of intelligence is independent from the g factor, or general intelligence factor, a construct developed to identify the common core of all cognitive tasks (Jensen 1978). It is important to note that the RCPM is a non-verbal test of cognitive skills, while the educational programme being evaluated is closely linked to verbal reasoning, much like logic itself. The intervention therefore does not train students to complete the RCPM, and thus any changes in RCPM score after the intervention will represent a genuine change in cognitive skills. All the questions were equally weighted in both tests.

8.5.2. Data

We included all students from the intervention and control classes who completed both the pre-intervention and post-intervention tests. After excluding students who scored the maximum in the first test—and therefore could not show improvement—our final dataset was made up of 18 students in either class, whose characteristics are summarised in the following table:

	Mean RCPM score	Males (%)	Non-Italian origin (%)
Control (N=18)	10.3	9 (50%)	5 (28%)
Intervention (N=18)	10.2	11 (61%)	5 (28%)

Table 8.1.: Sample data.

Here, we have included all observable variables that may affect the rate of improvement in the tests used. As the students were all of the same age, we performed balancing tests for sex, nationality of origin, and initial RCPM score. Given the relationship between learning and intelligence outlined by Jensen (2006), it is possible that initial cognitive ability can affect the rate of learning of both mathematical literacy and cognitive skills. Furthermore, according to Vaci, Edelsbrunner, Stern, et al. (2019), the benefits of practice increase with intelligence, suggesting that a child with higher initial cognitive skills would be able to improve their mathematical literacy more than their peers just from the standard math classes. We therefore included the initial RCPM score in the relevant characteristics.

8.5.3. Statistical analysis

To assess the randomness of class compositions, we used a Student's t-test to compare the mean values of each covariate between the two groups; we chose this statistic because the variance of each covariate was similar between groups, and Student's t-test is appropriate for very small samples (Winter 2013).

The results are showed in Table 8.2.

The results show no significant difference between the two groups, validating the assumption of the class compositions being as good as random. We therefore ran two unadjusted regressions with score difference as the independent variable and intervention group as the dependent dummy variable, for both mathematical literacy and cognitive skills. In this analysis, the regression coefficient of the intervention

	RCPM score	Sex	Non-Italian origin
t statistic	-0.23	1.07	0
p-value	0.42	0.62	1

Table 8.2.: Results of the t-test.

variable represents the mean difference in the outcome between the intervention and control groups. Since the sample was relatively small, we set the significance level at $p = 0.1$. An important issue that we were unable to adjust for is the potential influence of a memory effect. As the post-intervention RCPM test was composed of the same questions as the pre-intervention test, our results may be biased by this effect, even if the students were not given solutions to the tests.

8.5.4. Results

The results are shown in Table 8.3.

	Cognitive skills	Mathematical literacy
Coefficient	0.55	2.27
Standard error	0.29	0.88
p-value	0.06	0.09

Table 8.3.: Results of the analysis.

We observed a marginally significant effect ($p < 0.1$), of the intervention on both cognitive skill and mathematical literacy. Notably, the regression coefficient for cognitive skills is smaller than that for mathematical literacy, indicating a smaller difference in scores between the two groups, but the smaller p value indicates it approached acceptable levels of statistical significance ($p = 0.05$). This result may be due to memory effect bias and the small sample size, as discussed above. However, there is no reason to suppose that memory effect bias is more prominent in one group than in the other, so this issue does not invalidate our findings. The RCT performed showed that the educational programme proposed helps to develop mathematical literacy and general cognitive skills, specifically fluid intelligence, stimulating the formation of mental models that support the continued development of skills and abilities throughout the various stages of education. As argued in the theoretical framework, the improvement that takes place stems from a variety of sources: reflection, analysis of error, familiarity with logical symbolism, development of thought at the metalevel via in-depth study of the relationship between syntax and semantics, stimulation of proceptual thought, and learning about logical connectives and their use.

9. Lovleis Educational Path

The program *Lovleis*[1] alternates between two distinct phases: a narrative phase, where a story featuring the class as the protagonist is read to the students, a playful phase, in which two-player games stemming from the story are explored. These phases interchange, shifting back and forth, with the story being linear and non-interactive, and the gaming phase being non-linear and interactive (Göbel, Mehm, Radke, et al. 2009).

The central focus of the Lovleis program is the analysis of strategies. Therefore, let us recall the suggestions we got from the game $\mathbf{T}\mathscr{U}\mathscr{V}\mathscr{A}$ regarding the concept of proof. A proof is a winning strategy for the Proponent in an asymmetric game between two players (Proponent and Opponent), where the Proponent wins when the game is finite, and the Opponent wins if the game is infinite. Key concepts in the "game of proving" include, among others, asymmetry, the absence of a draw, and the idea of winning when the play is not finite. The Lovleis program was developed considering both these aspects related to proof and the fundamental notions of game theory.

9.1. Background and Related Works

Storytelling, combined with gaming and aimed at learning, is now a characteristic of many digital games, a trait shared with the Lovleis program. In Lovleis, the narrative is enhanced by evocative images projected on the IWB (Interactive Whiteboard), which are essential for catalyzing and stimulating the students' imagination. To progress in the story, it is necessary to achieve a specific goal (Schell 2008). Each chapter of the story corresponds to a distinct environment, encouraging various skills and abilities.

The tasks that interrupt the narrative consist of two-player games, to be conducted in class, between students or—in some cases—with the teacher.

Regardless of our general theoretical framework, which emphasizes the importance of analyzing strategies in two-player games as precursors to mathematical proofs, the introduction of games into the mathematics curriculum is supported by strong pedagogical reasons. Games indeed spark enthusiasm and participation: students are engaged, immerse themselves in the activity, discuss solutions, and analyze different strategies, as also highlighted by Ernest (1986). The game allows teachers to put students in problem-solving situations, functional to the ability to develop strategies and to unleash potential that students are sometimes unaware of and that are seldom highlighted in standardized situations. This approach has a positive impact not

[1] The name is a tribute to Ada Lovelace.

only cognitively but also socially, emotionally, and linguistically (Ferri, Matteo, and Pellegrini 2023).

The games in this educational path require mathematical reasoning and a significant capacity for abstraction in their execution or strategy. Additionally, these games naturally and healthily channel the competitive spirit of the students (Salomone 1979).

In Lovleis, the narrative in which the class is the protagonist creates an immersive learning environment. Despite its fantastical setting, it becomes familiar to the students, acting as a *raison d'être* for the games they must participate in. As reported in (Naul and Liu 2019), literature has shown that immersive learning environments, like educational digital games, often integrate narrative elements into their design. This is because storytelling can make learning more meaningful for students.

9.2. Description of the path

In outlining the path, we will limit the description of the narrative component to what is strictly necessary to understand the games presented[2]. In the story, the class is trapped in a fantastical world and must find a way out. On their journey to get back home, they pass through various cities and villages, each inhabited by people who play a particular two-player game, or multiple two-player games that are closely related to each other.

9.2.1. First Activity: Tic-Tac-Toe

The first game introduced is Tic-Tac-Toe. In the story, after ending up in the fantastical world, the class arrives in the city of *Tictacto*, where Tic-Tac-Toe is not just a pastime, but a way of life. After reviewing the rules in class and playing a few games, the students participate in a Tic-Tac-Toe tournament that takes place in the city. This allows each student to confirm their understanding of the game's rules and begin to develop implicit strategies.

We next proceed to the explicit analysis of strategies in Tic-Tac-Toe by means of a *escamotage*: the story introduces Timoteo, a resident of Tictacto, who has a problem—he can never win against his sister Celeste, who always starts first by placing her "X" in the center of the grid. To simplify the class's analysis, the focus is narrowed to the subset of games that begin with an "X" in the center. Timoteo, having seen the class's excellent performance in the Tic-Tac-Toe tournament, asks them for help. The students are then encouraged to write a consoling letter to Timoteo, offering him suggestions and strategies to avoid losing to his sister. Through this activity, students begin to engage with and explore the concept of strategy, laying the groundwork for future lessons. This exercise also brings to light various difficulties that are comparable

[2]The full story can be found in Italian at `www.oiler.education/scuola/materiali/primaria/lovleis`

to those encountered by students during a mathematical proof, as will be discussed in the following section.

In the subsequent discussion about the various letters written to Timoteo, two key aspects of strategy are highlighted and brought to the fore. The first fundamental aspect of a strategy is that it is developed *a priori*, meaning before the game begins. From a computational perspective, this is akin to a program that provides the player with guidance on what moves to make in response to the opponent's moves. The second essential element is that the strategy must be as general as possible: since the player cannot know in advance what moves the opponent will make, the strategy must encompass all possibilities, being capable of countering every move the opponent makes. These two key aspects are also sources of significant difficulty, as we will analyze later.

The next step involves finding a representation of the concept of strategy that is clear to interpret and highlights both the sequentiality and generality of the strategy. As analyzed in previous chapters, the optimal representation is that of a tree. More specifically, we see the following definition[3]:

Definition 31 (Strategy Tree)**.** *Let G be a two-player game,* **P** *and* **O**. *A* **P***-position (resp.* **O***-position) is a position where* **P** *(resp.* **O***) has to move. A* **P***-strategy is a tree where in every* **P***-position there is exactly one outgoing branch, while in every* **O***-position there are many outgoing braches, each one for a possible moves the* **O** *can make.*

In Figure 9.1, Celeste has the blue "X" and Timoteo the red "O". Timoteo's moves are indicated with a red arrow, while Celeste's are marked with a blue arrow.

[3] The definition, as you can see, differs from the one given in chapter 1. We believe that the distinction between partial and total strategies is overly complicated and artificial for non-university students.

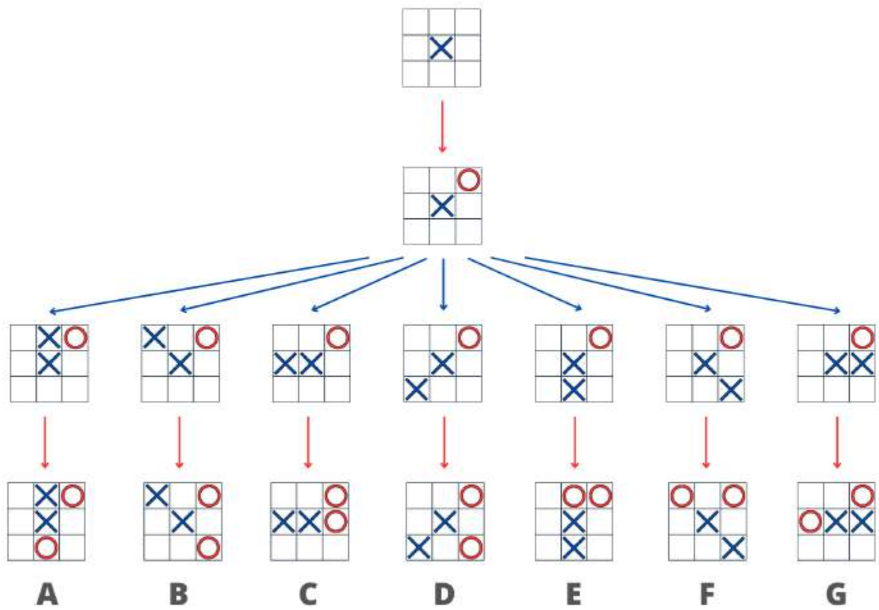


Figure 9.1.: A partial strategy for Timoteo. To complete it, all the sub-trees A, B, C, D, E, F should be completed.

As can be seen, when it's Timoteo's turn, there is a single red arrow indicating the move that Timoteo must make. On the other hand, when it's Celeste's turn, all the moves that Celeste could make during an actual game are considered. Therefore, Timoteo's strategy is capable of responding to any move made by Celeste.

We conclude by stating that a strategy is considered a winning strategy if it guarantees the victory of player P who follows it, that is, if every leaf of the tree is a winning position for P. The strategy we provided for Timoteo is not winning because, in some cases, it leads to a draw; however, it never results in a loss.

Let us notice how, also on this occasion, the proposed paths are interconnected: if one has already followed Zermelo's path, it is possible to further analyze the concept of strategy, formally linking it to quantifiers. A game between two players P and O unfolds in the following way.

Move 1 by O
Move 1 by P
Move 2 by O
Move 2 by P
Move 3 by O
...
P wins

Replacing P wins with $\neg O$ wins in the case of a drawing strategy.

9.2.2. Second Activity: Anti-Tic-Tac-Toe

Continuing the journey through the city of Tictacto, the second activity takes the class to the *Knaves' District*. Even the knaves are fans of the game of Tic Tac Toe, but with an intriguing twist: *the winner loses*. This variant, called *Anti-Tic-Tac-Toe*, brings two interesting aspects. First, it shows the class how slightly modifying the rules of a game (like reversing the winning conditions) can completely change its dynamics. Secondly, we find Anti-Tic-Tac-Toe more interesting than normal Tic Tac Toe because, for the first player, there is only one strategy to avoid defeat. To make the experience even more engaging and meaningful, the teacher assumes the role of the knaves. These characters will not allow the class to proceed until the students manage to tie at least one game. The better the teacher plays as the second player of Anti-Tic-Tac-Toe, the more intriguing the game will be for the students. In the future, one might consider a software that plays the best strategy.

The only way a player can avoid losing when making the first move is to play as follows: place their "X" in the center and then proceed symmetrically (with respect to the center) to the "O" of the knaves. Intuitively, in this way, it is impossible for the first player to make 3 in a row because the second player would make it first.

Anti-Tic-Tac-Toe provides a fundamental insight: even though a strategy can be exhaustively expressed through a decision tree, in this case representing the strategy through its central idea of symmetry turns out to be more effective. In other words, a player who blindly follows the strategy provided by the tree may end up drawing without realizing the crucial role played by symmetry and, most importantly, why this strategy actually works. Verbally expressing the strategy—namely, placing one's "X" in the center and then mirroring the opponent's moves symmetrically around the center—conveys the underlying idea more directly and makes the strategy more intuitive and understandable. This observation is crucial to make in class: in other words—and more deeply—it highlights that a derivation tree is not sufficient if not accompanied by an explanation of the ideas that led to that derivation. Expressing the key idea behind a proof is part of those social rules that need to be integrated with derivation rules to achieve a proof, as in Chapter 6. For centuries, as discussed in Chapter 2, the "brilliant idea" has indeed been the only interesting factor in a proof.

9.2.3. Third Activity: Pick15

In the third chapter of the journey, the class enters the Pick15 neighborhood, where the inhabitants play a different two-player game: *Pick15*.

In Pick15, all the numbers from 1 to 9 are placed on the table. Each player, in turn, chooses a number and adds it to their collection. If a player manages to have three numbers whose sum is exactly 15, they win. If, at the end of the challenge, no one has achieved the goal, the game is a draw.

After the class has played several games to grasp the dynamics of the game, one moves on to the central part of the activity: *understanding why the Pick15 neighborhood is located in the city of Tictacto*. Narratively, the need to unravel the mystery of

Pick15 is linked to finding the password to enter the Morris Tower, where they will meet the regent Trissa, a figure who will help the class get back home.

Initially, the class is divided into small groups to gradually bring out the similarities between the two games. In particular, some aspects to touch on are the following:

- Both games are played between two players;
- in both games, there is a draw when neither of the two players manages to win;
- in both games, the maximum number of moves is 9;
- in both games, winning involves using three of one's symbols or numbers that satisfies a certain property (arranged in a row in the case of Tic-Tac-Toe and summing up to 15 in the case of Pick15).
- in both games, there is the concept of a "double play": a particular configuration where no matter what move the opponent makes, one can win in the next move. In other terms, there is a strong resemblance between the two typical *valid arguments* of the games.

Once these key points are noted, the discussion can potentially continue to discover the deeper analogy: Tic-Tac-Toe and Pick15 are different representations of the same game. The point of contact is represented by the magic square shown in Figure 9.2.

2	7	6
9	5	1
4	3	8

Figure 9.2.: The sum of each row, column, and diagonal is always 15.

Indeed, taking a number in Pick15 and adding it to one's own numbers is equivalent to placing one's own symbol ("X" or "O") in the corresponding cell of that number in the magic square. Furthermore, having a sum of 15 among one's numbers is equivalent to having 3 in a row. The interested reader can analyze in detail all aspects of the isomorphism.

To make the analogy between the two games even clearer, two students can be asked to play against each other: one playing Tic-Tac-Toe and the other playing Pick15, with a third student acting as an interpreter.

9.2.4. Fourth Activity: Three Men's Morris

In the the last chapter of the narrative set in the city of Tictacto, the game of Three Men's Morris is introduced. Narratively, the game is played inside the Morris Tower, and winning a tournament is necessary to meet the regent Trissa, who will provide the class with directions on how to return home.

Three Men's Morris is played on a board like the one shown in Figure 9.3. Each player has three pieces, and the objective of the game is to align one's three pieces vertically, horizontally, or diagonally (just like in the game of Tic-Tac-Toe).

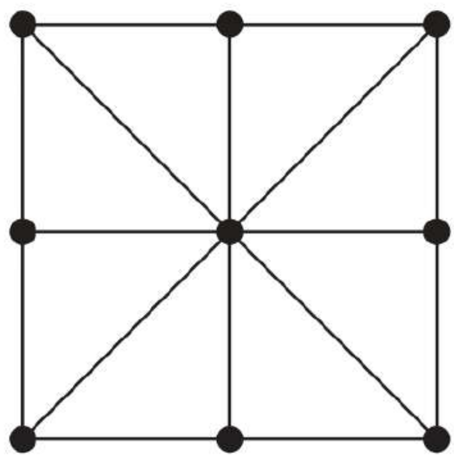
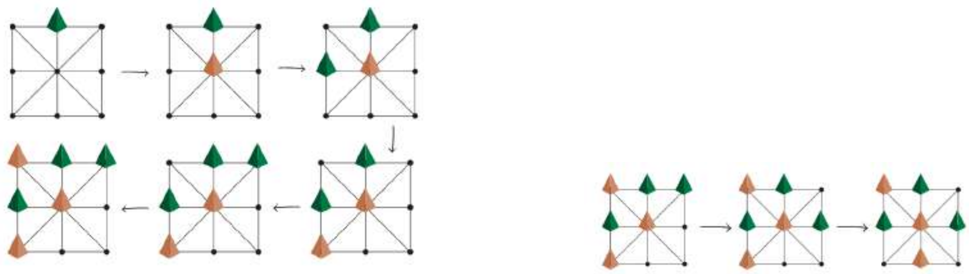


Figure 9.3.: The game board of Three Men's Morris.

Taking turns, each player places their piece on a point of their choice. Referring to the following Figure 9.4a, the green player starts and the game proceeds in turns.



(a) First phase of the game: players arrange their pieces as in Tic-Tac-Toe.

(b) Second phase of the game: players move their pieces along the lines.

When both players have placed all three of their pieces, unless there is already a winner, the second phase begins. In the second phase, on each turn, a player moves one of their pieces to a nearby point (*i.e.*, connected by a line) that is not occupied by another piece, see Figure 9.4b. This continues until a player achieves 3 in a raw.

Now let's delve into the choice of including Three Men's Morris in the educational path. This game is an alignment game that fits well into the city of Tictacto and introduces significant variations that enrich its theoretical complexity. First of all, Three Men's Morris allows for the movement of pieces during the game, a feature not present in either classic Tic-Tac-Toe or its misère variant. One of the most important peculiarities that differentiates it from the others discussed so far is its duration: a game can, in fact, **continue indefinitely**. This can happen, for example, if both players continuously repeat the same moves. In the face of this possibility, it is of fundamental importance to discuss with the class the following question: *who wins if the game goes on for a long time with the players continuing to make the same moves?*. The desired conclusion is that it wouldn't make sense to declare one of the two players the winner: if the game goes on indefinitely, then a draw is declared. Thus, we begin to understand that an infinitely long game can be assigned with winning conditions: as we will see later in games, and as we have already seen with the game $\mathbf{T}\mathscr{UVA}$, a draw is not the only possibility. In some games, if a play goes on indefinitely, it makes more sense to declare one of the two players the winner.

9.2.5. Fifth Activity: Leva Leva

Narratively, when the class wins the tournament at Morris Tower, they meet the regent Trissa who explains them, in addition to the reason for their journey, which path to follow and whom to approach to return home. The class is directed towards the country of Leva Leva[4], where they play a category of two-player games that all have one characteristic in common: initially, there are n objects on a table, and each of the two players can remove—taking turns—a certain number of objects while respecting some rules. The player who takes the last object either wins or loses, depending on the game. The initially proposed game is called *Leva* 5[5].

[4] In Italian, "Leva" means "Take" or "Remove".

[5] A notable use of this game in the classroom is analyzed in the introduction of (Brousseau 1997).

In this variant, 20 pieces of paper are placed on the table. Each turn, a player removes—at their choice—from a minimum of 1 to a maximum of 4 pieces of paper. The player who cannot take a piece of paper from the table anymore loses; in other words, the winner is the one who removes the last piece or the last pieces of paper from the table.

For the first time in Lovleis, the class plays a game where drawing is not an option. This distinction from previous games should be emphasized: in some games, the only outcomes are a win for one of the two players, with no possibility of a draw (clearly Leva 5 cannot continue indefinitely because the number of pieces on the board strictly decreases with each turn). After explaining the rules to the class and letting them play a few rounds, the question arises whether there is a winning strategy for either player. Although it will not be shared with the class, the reader might find of their interest that the existence of a winning strategy is guaranteed by Zermelo's theorem: in a finite two-player game of perfect information, which cannot end in a draw and where chance plays no role, one of the players has a winning strategy.

In this case as well, a tree diagram could be suggested but only after the students have grasped the underlying idea, because the winning strategy is based on arithmetic reasoning: *understanding that if the second player manages to keep the number congruent to 0 modulo 5, they will win the game*.

As already mentioned, the game of Leva Leva lends itself well to discussions about game variants: variations can be made in the number of objects initially on the table, how many can be removed each turn, and the winning conditions. With advanced grades, the conversation can be deepened by noting that—if a winning strategy is found for each variant—one can have a *set* of winning strategies: in a sense, a meta-strategy that determines, based on the number of objects on the table, how many pieces can be removed, and who wins (whether it's the player taking the last piece or not), which of the two players has a winning strategy and what that strategy is.

As will be discussed in the next section, the strategy for this type of game is usually developed from a concatenation of *valid arguments*.

9.2.6. Sixth Activity: Mountain Nim

In the story, the class is guided towards Mount Nim (Figure 9.5), where an annual celebration is about to take place, featuring the famous game of Nim. The game of Nim, indeed, is always a variation on the games discussed in the previous section: in Nim, the rule for removing objects from the table is not about their quantity but their geometric arrangement. Players can remove as many objects as they wish, but only from one of the rows at a time.

Figure 9.5.: One of the images that is projected in the classroom during the reading of the story, representing Mountain Nim.

Various initial dispositions are analyzed, with the aim of finding a winning strategy for each.

In this case, the winning strategy is too complex to be studied in primary school (a good understanding of numbers written in base 2 is required). An attempt to cover it might be feasible in middle school and certainly in high school.

As can be seen, the game of Nim doesn't hold much value within the Lovleis program if the intention is merely to lead the class to a demonstration. However, it's believed that a game like Nim, rich both theoretically and strategically, has significance not only cognitively but also culturally.

9.2.7. Seventh Activity: Soldier Game

"[Soldier Game] combines extreme simplicity
with extraordinary strategic subtlety"
Martin Gardner

The class is then guided by an citizen of Leva Leva towards the village of Ghisa, where the journey will conclude and the students will be able to return home. Now, the class will encounter two different games, one known as the Soldier Game and another created *ad hoc* to simulate a demonstration in propositional logic and first-order pure logic.

In the narrative, three soldiers block the road leading to the village of Ghisa and refuse to let the class pass. The game is part of the family of the famous Hare Games: two-player games that were popular in Medieval Europe up until the 19th century.

The game is played on a chessboard like the one in Figure 9.6, where the blue dot represents a student who must escape the soldiers, while the three red dots are the three soldiers.

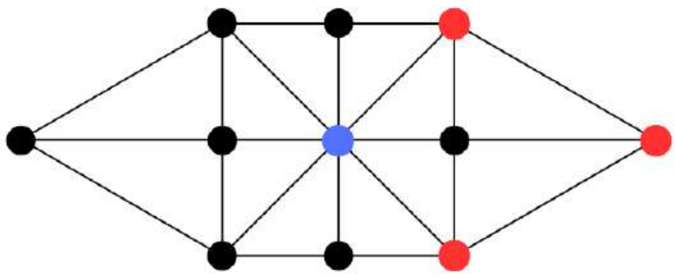


Figure 9.6.: Soldier Game chessboard.

One player moves the blue piece and the other moves the three red pieces. Players take turns making a move along an arc of the graph. The student can move along any arc of the graph, while the three soldiers can only move "forward", meaning from right to left, or up and down. The soldiers' goal is to capture the student, which means leaving them with no possibility of movement, by reaching one of the following three configurations 9.7.

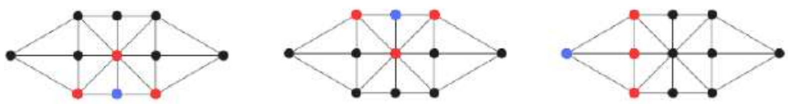


Figure 9.7.: The three cases in which the red (soldiers) win.

The student's goal, on the other hand, is to "escape" from the soldiers without being captured. Since the soldiers cannot move backwards, if the student passes the vertical line of the last soldier, then they win. In other words, the student wins if they reach the circle located at the far right.

The nature of the game is complex, and in the class, the focus will be more on theoretical observations rather than an in-depth analysis of the strategy. First of all, it is an asymmetric game, where the rules that the two players must follow and their respective goals are different. It is the first case of an asymmetric game that the class encounters in Lovleis.

Furthermore, in the Soldier Game, there is an interesting fact: as in the case of Three Men's Morris, a play can continue indefinitely, potentially with players repeating the same moves. In Three Men's Morris, it was seen that the only sensible rule in this eventuality is to declare a draw between the two players. The class reflects together on what would be reasonable to decide in the Soldier Game: given that the blue piece is escaping from the soldiers, it is more logical to say that—if the game goes on indefinitely—the blue wins. As a result, the soldiers have no interest in wasting time repeating moves, as this would lead to their loss.

We underline here that the study of asymmetric games with infinite victory conditions is not new to class proposals; see, for example, (Antoine, Beffara, Molinier, et al. 2022).

9.2.8. Eighth Activity: Ghisa Village

Before exploring the last game, which is a graphical transposition of the game $\mathbf{T}\mathscr{UVA}$, let us take a step back and recall what was said about the proof at the beginning of the chapter, in light of the games just presented.

The *Game of Proving* (where a proof is a winning strategy) is a two-player asymmetric game with perfect and complete information, where the concept of infinity and conditions imposed on it play a fundamental role.

This aspects where all tackled by Lovleis' games: all the games in Lovleis are two-player games, and we focused particularly on the concepts of victory, defeat, and draw; of finite and possibly infinite games; of symmetric and asymmetric games. Strategies and their possible representations were also carefully explored.

Just like the proof, the final game we will present–*the Sacred Formulas of the Geese*–is an asymmetric game in which there is no draw. One player wins if the game ends and the other wins if the game goes on indefinitely. It is a game where both players play on the same formula, meaning—in practical terms—they move the same piece.

In the narrative, after the class successfully overcomes the threat of the soldiers, they reach the Village of Ghisa. The inhabitants of the Village of Ghisa are wise geese. The geese explain to the class that in order to return home, they must make the *Great Leap*, and to do so, they must train with the "sacred formulas". Each sacred formula contains a game between two players, and the class's task is to understand which of the two players has a winning strategy and what that strategy is. There are a total of 13 formulas, and each sacred formula corresponds to a $\mathbf{T}\mathscr{UVA}$ game played on a specific propositional logic formula. In particular, the formulas are the same as those of propositional logic found on the platform *Luì*, discussed in chapter 5. How to construct a sacred formula of the geese from a propositional logic formula will be discussed later.

The game is played between two players, named **P** and **O**, who play on a graph, both moving the same pawn in turns. To start the game, the pawn is placed on the dot with wings. **O** begins and, on each turn, the pawn must be moved along a direction indicated by arrows. Whenever **O** passes over a colored arrow, the color is unlocked and added to the list of activated colors. The player **P** can only move over an arrow if its color has been previously activated by **O** (except for black, which is always considered active[6]). If **O** can no longer move, **P** wins; otherwise, if the game continues indefinitely, **O** wins. The structure of the graphs excludes the possibility that **P** can no longer move: in other words, $F \to \perp \in \mathscr{U}$.

Let us look at an example with formula number 8 (see Figure 9.8). **O** starts and the only move they can make is to go down following the black arrow. **P** can now choose either to return to the starting point (thus restarting the game from the beginning) or to go down to the left; indeed, **P** cannot go down to the right because the green color has not been activated by **O**. **O** can now only choose the green arrow, activating the color. If the game restarts, **P** can now go to the green on the right, winning the game.

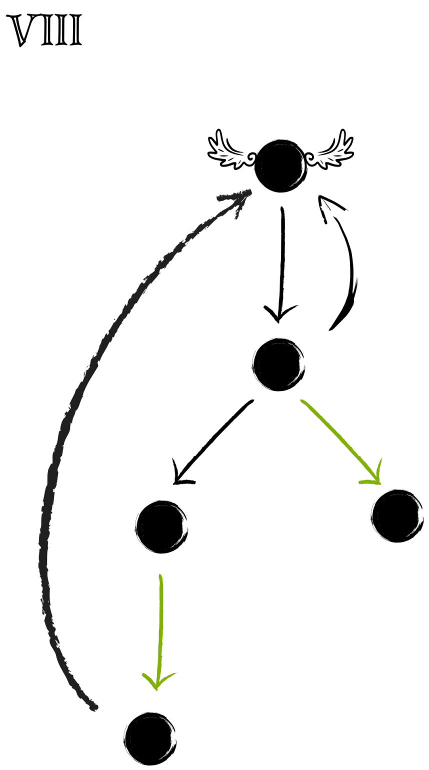


Figure 9.8.: The eighth geese's sacred formula, which corresponds to the formula $G, G \to \perp \to \perp$.

As an additional example, let us look at the XII formula, as shown in Figure 9.9.

[6]Black plays the role of $\perp$.

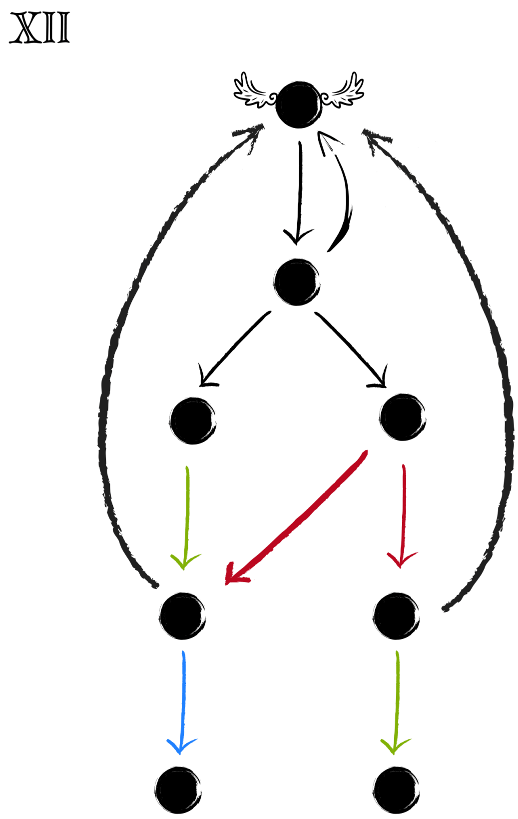


Figure 9.9.: The penultimate sacred formula of the geese, which corresponds to the formula $(B \to G) \to \bot, B \to R, (G \to R) \to \bot \to \bot$.

In this case, it is **O** who has a winning strategy. The reader can verify that **O** can always force **P** to restart the game. However, it is worth noting that if **O** plays a non optimal strategy, such as activating the green and then following the red arrow downwards instead of diagonally, **O** would lose the game.

In general, the formulas have been designed such that sometimes there is a winning strategy for **P** and other times for **O**. Furthermore, some formulas are constructed in a way that allows one player to win even if the other has a winning strategy, provided the latter plays non optimally. This approach highlights once again the difference between and winning a single play and actually having a winning strategy .

It should be noted that many of these formulas do not give rise to games with a wide range of choices. However, this is not a problem, because the focus for the class is on finding a winning strategy rather than winning a single game.

9.2.8.1. The Sacred Formulas of the Geese and the $\mathbf{T}\mathscr{UVA}$ Game

The game of the geese faithfully reproduces what happens with the game $\mathbf{T}\mathscr{UVA}$ for propositional logic. Let us look in detail at how to construct the game graph for the game of the geese starting from a normal formula $F_1, \ldots, F_n \to A$. Firstly, each propositional letter in the formula is assigned a color; the $\perp$ is by default associated with the color black. Thus, the game graph can be constructed inductively. Initially, there is only the dot with wings, shown for simplicity in Figure 9.10 as a white dot.

Figure 9.10.: Initialization of the game graph.

Let us assume, for instance, that the propositional letter A has been assigned the color green. Then, from the node with wings, as shown in Figure 9.11, a green arc originates that leads to another node. From that node, n arcs will then branch out, one for each subformula F_i. Clearly, the arcs leaving the node will not all be black, but will be colored according to the conclusion of each F_i.

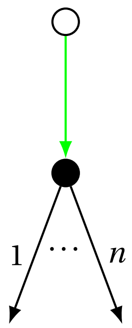


Figure 9.11.: Creation of the second node and of the related arcs.

To conclude this section where the educational path has been presented, we provide the summary Table 9.1 outlining, for each game, the main concepts discussed.

Table 9.1.: Summary of Lovleis educational path

Two-Player Game	Transposed Concepts
Tic-tac-toe	Finite game; Strategies as graphs
Anti-tic-tac-toe	Negation of winning condition
Pick15	Isomorphism
Three Men's Morris	Infinity; Stalemate at infinity
Leva 5 (race to 20)	Arithmetic strategy
Nim	Case analysis; Complex arithmetic strategy
The Soldier's Game	Asymmetry; Winning at infinity; No draw
The Goose Game	Faithful transposition of the propositional $\mathscr{TUVA}$ Game

9.3. Analysis of the Classroom Experience

Our analysis is focused on what appears most relevant among the findings from the trials.

9.3.1. Tictacto

We aim to investigate whether some typical difficulties students face in mathematical proof can also be found in researching and communicating a strategy in a two-player game. Since it's possible—at a school level—to work on strategies much earlier than on proofs, we wonder whether addressing the challenges that arise around strategies might indirectly tackle the difficulties associated with proof.

To do so, we return to the intuitive definition of a strategy for a player **P** in a game, which is a function that—given a position—tells **P** what move to make. Specifically, having a winning strategy for **P** means satisfying the following formula:

$$\forall \mathbf{O}_{move} \exists \mathbf{P}_{move} \forall \mathbf{O}_{move} \exists \mathbf{P}_{move} \ldots (\mathbf{P}\text{wins})$$

. Similarly, having a non-losing strategy is equivalent to satisfying

$$\forall \mathbf{O}_{move} \exists \mathbf{P}_{move} \forall \mathbf{O}_{move} \exists \mathbf{P}_{move} \ldots (\neg \mathbf{O}\text{wins})$$

.

Let us analyze the difficulties that emerged during the experimentation conducted with third and fourth grade primary students. As seen in the previous section, during the first activity, students are asked to help Timoteo not lose against his sister Celeste in the game of Tic-Tac-Toe. To do this, they must write him a letter explaining how to behave.

Almost all pairs of students first struggled with effectively communicating the *sequentiality* of the strategy; see the example shown in Figure 9.12.

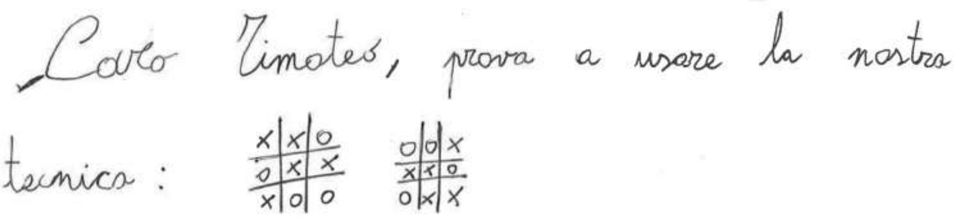


Figure 9.12.: The students only show two final positions where the game ends in a draw, without indicating the sequence of moves that led to these final positions.

Indeed, by looking only at the final position of a play, one cannot know the order in which the moves were made. In other words, by looking at the final position only, Timoteo can't know which move to make at what time. After discussing the problem with the students, a solution many found was to add numbers as indices of the moves made, to indicate at what point in the play each move is executed, as shown in Figure 9.13.

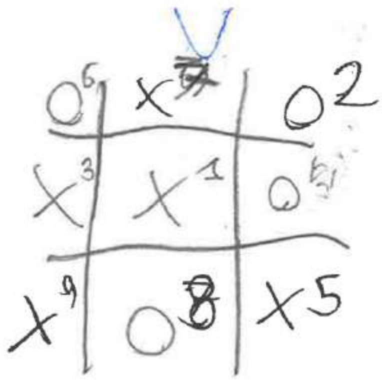


Figure 9.13.: The students added indices to indicate the sequence of moves in the game.

At this stage, a crucial issue—previously implicit—becomes explicit: *how can one be sure of the move Celeste will make?*. Many students overlooked the generality of the strategy, believing they could predict Celeste's moves to guide Timoteo's strategy, as seen in Figure 9.14.

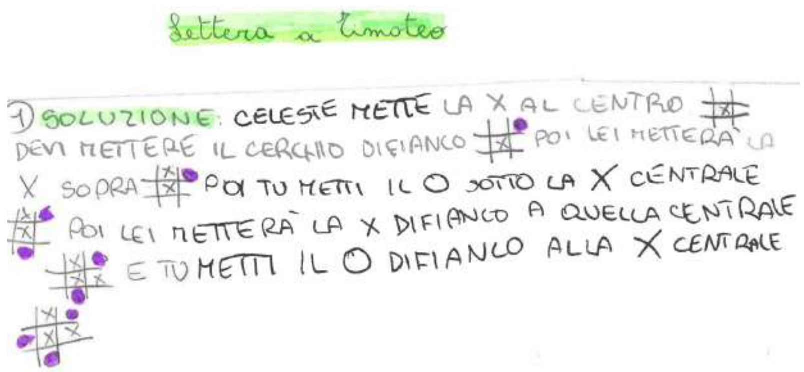


Figure 9.14.: The student treats Timoteo and Celeste as equivalent.

In other words, most students had difficulty considering *all* possibilities, that is, all the potential moves Celeste could make at each turn. Showing just one scenario where Timoteo manages to win or draw is not enough: referring back to the previously discussed formulas, satisfying the formula $\exists \mathbf{O}_{move} \exists \mathbf{P}_{move} \exists \mathbf{O}_{move} \exists \mathbf{P}_{move} \ldots (\neg \mathbf{O}\text{wins})$ is insufficient. Instead, one must satisfy and adequately communicate the formula $\forall \mathbf{O}_{move} \exists \mathbf{P}_{move} \forall \mathbf{O}_{move} \exists \mathbf{P}_{move} \ldots (\neg \mathbf{O}\text{wins})$. Other students, although understanding that their approach lacked generality, were unable to do anything but address specific plays—*i.e.*, examples—as shown in Figure 9.15. In their report, students also provides what has been termed a *valid argument*, namely the well-known "fork". If one manages to set up a fork, they can win in their next turn.

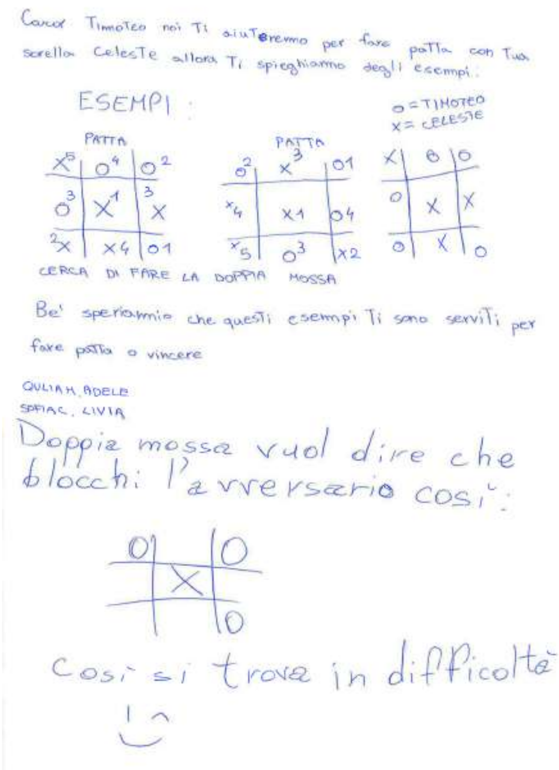


Figure 9.15.: The students provide several examples, and then present a *valid argument*.

In these cases, where it seems clear that Celeste's move cannot be predicted, the approach is limited to providing examples, thus losing generality. In other words, the quantifier on **O**'s moves can be seen, in a sense, as a hybrid concept between an existential quantifier and a universal quantifier. See 9.16 for reference.

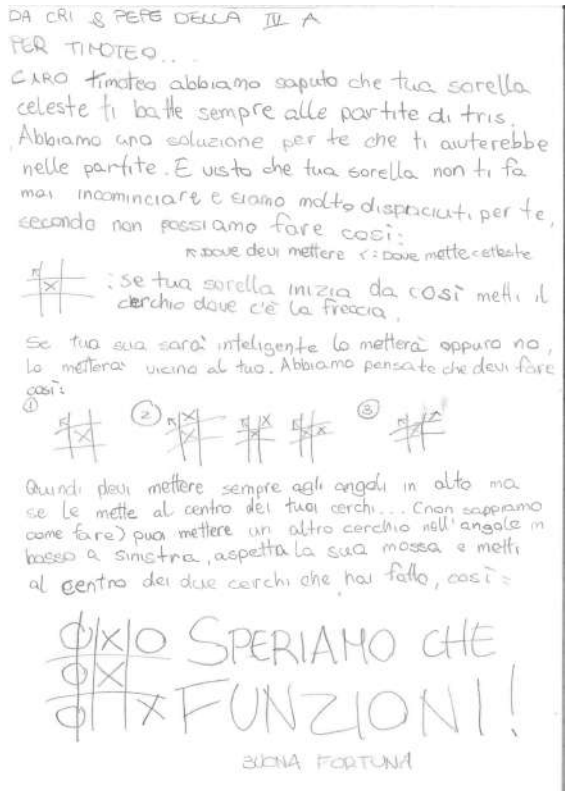

DA CRI & PEPE DELLA IV A
PER TIMOTEO..
CARO timoteo abbiamo saputo che tua sorella celeste ti batte sempre alle partite di tris. Abbiamo una soluzione per te che ti aiuterebbe nelle partite. E visto che tua sorella non ti fa mai incominciare e siamo molto dispiaciuti per te, secondo non possiamo fare così:
↖: dove devi mettere ✕: dove mette celeste
: se tua sorella inizia da così metti il cerchio dove c'è la freccia.
Se tua sua sarà inteligente lo metterà oppure no, lo metterà vicino al tuo. Abbiamo pensato che devi fare così:
① ② ③
Quindi devi mettere sempre agli angoli in alto ma se le mette al centro dei tuoi cerchi... (non sappiamo come fare) puoi mettere un altro cerchio nell'angolo in basso a sinistra, aspetta la sua mossa e metti al centro dei due cerchi che hai fatto, così:
SPERIAMO CHE FUNZIONI!
BUONA FORTUNA

Figure 9.16.: The student analyzes some paths of the strategy tree, clearly not exhausting all possibilities.

Therefore, the goal was to encourage the class towards as general a strategy as possible, seeking suitable forms of strategy representation. Particular attention was paid to the representation using a tree. The students then completed the tree graph, albeit partially.

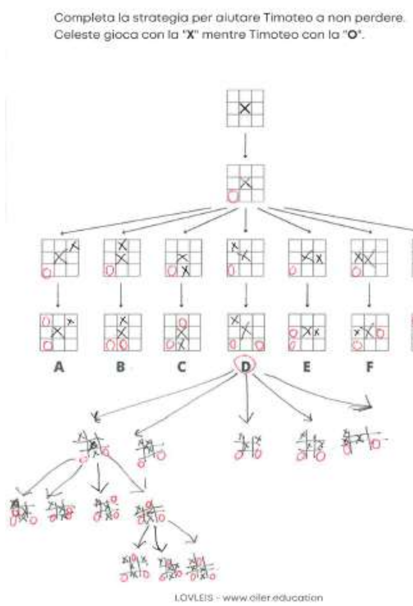


Figure 9.17.: A strategy tree completed partially but correctly. The students were asked to complete it autonomously only on the case D.

In summary, two profound conceptual challenges that cause difficulties were identified, where the second becomes apparent only after addressing the first. The first challenge is understanding that in describing a strategy, it is essential to express the sequence of moves leading to a certain situation, emphasizing its relevance. The second challenge is realizing that a strategy must be general in nature, considering all possibilities, as one cannot predict the opponent's moves in advance. The workaround found by students who understood that Celeste's moves couldn't be predicted, but who still couldn't formulate a complete and satisfactory reasoning, was to provide examples or valid arguments, namely partial strategies.

Let us now compare the dynamics just discussed with the issues found in literature of students facing a proving. We begin by analyzing the interesting report by A. Selden and J. Selden (2015). Initially, a typical dynamic in literature is shown, illustrating how during a geometry course (Chazan 1993) some students, despite specific training on proof, confused empirical evidence with deductive proofs. Some indeed believed that empirical evidence was sufficient as proof. It is interesting to note here that, as discussed in chapter 5, most theorems primarily involve a universal quantifier, taking the form $\forall x P(x)$. More specifically, many statements in geometry are of the type $\forall \vec{x}(I_1(\vec{x}) \wedge \ldots \wedge I_n(\vec{x}) \rightarrow T(\vec{x}))$, meaning that if for a set of points all hypotheses I_1, ..., I_n are verified, then the thesis T is also verified. On the other hand, finding evidence for the theorem $\forall x P(x)$ instead means satisfying the formula $\exists x P(x)$. In geometry, evidence for the theorem $\forall \vec{x}(I_1(\vec{x}) \wedge \ldots \wedge I_n(\vec{x}) \rightarrow T(\vec{x}))$ is not so much given by satisfying the formula $\exists \vec{x}(I_1(\vec{x}) \wedge \ldots \wedge I_n(\vec{x}) \rightarrow T(\vec{x}))$ as by satisfying the formula $\exists \vec{x}(I_1 \wedge I_2 \wedge \ldots \wedge I_n \wedge T)$. In other words, by empirical evidence, we mean a case where the conditions are verified and indeed the thesis is also verified. Rarely is an example given to a geometry theorem by negating the hypotheses. The difficulty in understanding quantifiers is, as discussed in the previous paragraph, already present when talking about winning strategies at the primary school level. The examples attempt to serve as a surrogate for the lack of generality.

The report by Selden and Selden indeed continues by discussing how one source of students' difficulties in discerning the logical structure of theorems is a lack of understanding of the meaning of quantifiers and that their order matters. Undergraduate students often consider the effect of an interchange of existential and universal quantifiers to have no impact. This dynamic is present in what we have discussed, even under two different aspects: initially, in the first conceptual node, there is a failure to understand that it is important to communicate the sequence of a strategy and, secondly, a lack of meaning on quantifiers, where all quantifiers are read as existential. We are also firmly convinced that explicit work on quantifiers, linking them to their Dialogue Rules as done in Chapter 7, is essential to prevent these difficulties.

As reported in (Stylianides, Bieda, and Morselli 2016), one of the studies with a larger sample was that of (Morselli 2006), who conducted interviews with 47 university students and found that participants' argumentative processes could be classified into four profiles[7]:

- work exclusively through algebraic manipulation;
- short explorations with examples and shift to algebraic proof;
- extended explorations with examples leading to reasoning about the conjecture;
- unfocused explorations with examples.

She identified, in particular, that participants exhibiting argumentation habits categorized into the fourth profile were less successful than other students. This suggests that exploration with examples can be very productive for proving as long as the exploration is focused and purposeful. It's important to note that throughout Lovleis's educational path, the explorations are always meaningful: the students, while playing, have the central goal of winning, which greatly fosters the development of conjectures and arguments.

Similarly, Lin and Wu (2007) suggest that the features of given examples influence the kinds of generalizations that students make. In cases where students are asked to reason from examples to prove a given conjecture, it may support students' argumentation process if a range of examples are provided for their review or if students are encouraged to generate their own examples so that they can determine which features are variant under the given conditions.

To conclude, we note how one reason that university students find proof so difficult is that their experience with constructing proofs is typically limited to high school

[7]This is a discussion about a specific case of proofs in algebra.

geometry (Moore 1994). Lovleis's program aims precisely to provide a broader perspective on proof, emphasizing those fundamental logical aspects that are too often left unspoken, and which are necessary for a good understanding of what a proof is.

9.3.2. Leva 5

An interesting connection between proof and winning strategy was found in the analysis of what happened in the game of Leva 5. Students first begin to realize that if they can leave their opponent with 5 objects, then they will surely be able to leave 0 and thus win. This is—using the words of Chapter 6—a *valid argument*, meaning a partial strategy that leads to victory only in specific cases. Subsequently, students realize that if they can leave their opponent with 10 objects, then they will certainly be able to leave 5. In this way, the valid argument from before is expanded to a broader class of situations. Thus, students will understand that if they can leave 15 objects, then they will be able to leave 10. Therefore, from a concatenation, or rather an expansion, of valid arguments, the winning and complete strategy is obtained.

This analysis is fully aligned with the much broader and renowned analysis by Brousseau (1997) in his celebrated introduction to the Theory of Didactical Situations.

Conclusion

The purpose of this thesis is to develop mathematical structures that can serve as a basis for the analysis of mathematical reasoning. The first part is grounded in proof theory, with the study and development of interactive models of logic derived from the semantics of proofs and programming, whereas the second half turns to Mathematics Education, with the study of the process of validation and its contribution to the development of mathematical reasoning in students. The validity of a mathematical claim is fundamentally interactive in two separate ways. On one hand, the act of proving is primarily a social process, in the sense that it involves constructing a shared consensus: a fact is deemed established when a proof is produced and accepted as such by the community, starting from identified axioms and using modes of reasoning that are considered valid. This operation is applicable in both a research context and in classroom teaching. On the other hand, the very structure of a proof, if fully formalized, is a means to justify a claim by defeating any attempt at contradiction. This form of dialogue between a Proponent and an Opponent is the foundation of the interpretation of logic in various formal games. More specifically, the game $\mathbf{T}\mathscr{UVA}$ was introduced and presented, expanding on the game proposed by Krivine and Legrandgérard (2007). Subsequently, the correspondence—under certain conditions—between winning strategies in the game and proofs in LK was shown. As argued, LK is in fact one of the most suitable environments for elucidating the concept of proof. To delve deeper into the connection with games, the LK system was manipulated to obtain a new system, $\mathrm{LK}_{\mathrm{game}}$, which consists of only two rules, one reversible and one irreversible. In this transformation, extensive study and use were made of the properties of focusing—already utilized when addressing games—and of reversion, which, to our knowledge, had not yet been exploited in games. In discussing these topics, an effort was made to conciliate, where possible, the two main branches of proof theory: on one side, *cut elimination*, and on the other, *proof search*. It was then shown, in Chapter 5, that the $\mathbf{T}\mathscr{UVA}$ game was not merely a theoretical device useful for interpreting the proof process and the search for proof through abstract interaction, but was a game that—with the necessary adjustments—could be made not only concrete, but also integrated with the most widespread theories in both the logical and educational environments. At this stage, the question was raised whether the presented full-abstraction could prove useful in a Mathematics Education environment, both for interpreting the proof process and for suggesting meaningful activities in a classroom setting. Three educational paths were then outlined, the creation of which made extensive use of the insights suggested by full-abstraction, both in the general philosophy of the pathways and in the individual activities.

The main contributions of the initial chapters are of a strictly logical nature, as they provide a complete and formal treatment—as well as an expansion—of a game presented only in draft form, in a few pages, in 2007. Moreover, a connection was found with deep aspects of proof theory such as focusing and reversion, paying great attention to cut elimination and stability under cut elimination.

From the perspective of Mathematics Education, a model was proposed to interpret the proof process that, on one hand, does not abandon mathematical rigor and, on the other hand, fits well with research on proof in Mathematics Education. It becomes clear how argumentation and proof are intertwined aspects of the same process. Furthermore, this work highlights how dialogical exchange and argumentation can be considered foundational for the teaching of mathematics, and how these can be approached from the beginning of primary school through various meaningful transpositions, while still remaining faithful to the theoretical component.

Moreover, guided by the game $\mathbf{T}\mathscr{UVA}$, we decided to investigate how students approach strategies in a two-player game, with the significant finding that the issues younger students have in managing and communicating a strategy are almost identical to the issues older students have when faced with a proof. This also suggests how, through two-player games, it is possible to work on proofs and the difficulties they entail well before what is done in usual educational practice.

It must, however, be acknowledged that the educational analyses of the Zermelo and Bul pathways need to be enriched, seeking to place the debate within educational theories of proven relevance. Moreover, it is difficult to distinguish the contributions of the two games, Zermelo Game and Bul Game, in their respective pathways, an analysis that could prove interesting. Finally, the quantitative analysis of the Bul Game pathway is certainly unorthodox and foreign to the Mathematics Education environment.

The work presents various paths for development. First of all, the online implementation of λuì can be enriched and improved both by giving users the possibility to create their own formulas and theories to play with, and in terms of integration between proof search and artificial intelligence, as discussed in Chapter 5. Regarding the educational component, the development of the Lovleis Game is planned. The game aims to create a bridge between the educational pathways and the online implementation of λuì, with a substantial focus on argumentation.

In summary, the game $\mathbf{T}\mathscr{UVA}$ serves to build a bridge between proof theory—as seen as a branch of mathematical logic—and educational theories, providing a theoretical foundation for some of these. The game can thus also be helpful in the design of educational pathways, highlighting the key points of the underlying theory. The educational pathways presented here were indeed conceived under the guidance of the game: Zermelo introduces quantifiers and reasoning about quantifiers, while Bul introduces connectives and predicate calculus; in other words, the first two pathways aim to introduce the rules of the game and at the same time a language suitable for logical reasoning. Lovleis, the last pathway, consists of playing.

The work of this thesis adds another pillar of support to the view that logic deserves a greater role in the educational environment. The arguments presented here dismantle the outdated view of logic as an intransigent and limiting instrument that stifles creativity, and instead reveal it to be an environment of discovery, where opinions are not only welcome but essential.

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ANNEXES

A. How the λuì Code Works

The game code is ran by www.galua.cc and is written in Python. Since the code separately handles Propositional Logic and First Order Logic modes, we will present them distinctly. The code was written with Mattia Sanchioni.

A.1. Propositional Logic

A.1.1. Class Formula Definition

The first step is to define, using a Json, a hierarchical and recursive structure as a representation of the formula. Referring to a Krivine normal formula in the case of propositional logic, a formula $F_1, \ldots, F_n \to F_0$ is a dictionary that has three main keys: "predicate_letter", "num_of_hypos", and "hypothesis". "predicate_letter" is the conclusion F_0 (which can be $\perp$ or a letter for predicates), "num_of_hypos" is n, *i.e.*, the number of hypotheses, and "hypothesis" is in turn a dictionary that has n keys, where the value associated with the key i is the dictionary representation of F_i.

As an example, we present the dictionary representation of the formula $(B \to G) \to \perp, (G \to R) \to \perp \to \perp$.

```
{
    "formula": {
        "predicate_letter": "false",
        "num_of_hypos": 2,
        "hypothesis": {
            "1":


            {
                "predicate_letter": "false",
                "num_of_hypos": 1,
                "hypothesis": {
                    "1": {
                        "predicate_letter": "G",
                        "num_of_hypos": 1,
                        "hypothesis": {
                            "1": {
                                "predicate_letter": "B",
                                "num_of_hypos": 0,
                                "hypothesis": {}
                            }
                        }
                    }
                }
            },
```

```
            "2": {
                "predicate_letter": "false",
                "num_of_hypos": 1,
                "hypothesis": {
                    "1": {
                        "predicate_letter": "R",
                        "num_of_hypos": 1,
                        "hypothesis": {
                            "1": {
                                "predicate_letter": "G",
                                "num_of_hypos": 0,
                                "hypothesis": {}
                            }
                        }
                    }
                }
            }
        }
    },
    "creator": "Luigi",
    "formula_name": "11"
}
```

It’s worth noting that in the main formula’s Json, there are also the keys "creator" and "formula_name", which are necessary for displaying the formula only.

The Json representing a formula is then translated into a data structure to make it manipulable within the code. Specifically, the Formula class is defined as follows:

```
class Formula:
    def __init__(self, fid, hypo: List['Formula'], conclusion: PropositionalLetter
        self.formula_id = fid
        self.hypo = hypo
        self.conclusion = conclusion

        self.creator = creator
        self.formula_name = formula_name
```

The parameters that identify a formula, and thus are used to generate the associated object, are:

- "F_id" which is a unique identifier for the formula.

- "hypo" which is a list of Formula class objects, i.e., the hypotheses, which creates the recursion.
- "conclusion" which is a PropositionalLetter.
- "creator" which is a string, empty by default.
- "formula_name" which is a string, empty by default.

Specifically, PropositionalLetter is a class defined as follows:

```
class PropositionalLetter:
    def __init__(self, value: str):
        self.value = value
```

In other words, a PropositionalLetter is nothing more than its representation as a string. It is represented as a class so that various methods could be implemented to allow its use, including the comparison between two PropositionalLetters to determine if they are equal, and the HTML representation to be provided to the frontend.

A.1.2. Game Rules

The game rules are then defined. The game is a class that keeps track of the ongoing play.

```
class Game:
    def __init__(self, session: str, formula_filename: str, vs_pc: bool = True):
        self.session = session
        self.vs_pc = vs_pc

        self.turn = 0
        formula_folder = os.path.join(ASSETS_DIR, GAME_NAME, "formulas")
        self.formula_path = os.path.join(formula_folder, formula_filename + ".Json
        formula_Json = Json.load(open(self.formula_path))
        self.F: Formula = Formula.from_Json(formula_filename, formula_Json['formul

        self.U: List[Formula] = [Formula(self.F.formula_id, [self.F.copy()], botto
        self.V: List[Formula] = [self.F.copy()]
        self.A: List[PropositionalLetter] = [bottom]
```

Specifically:

- "session" indicates the game's identifier.
- "formula_filename" is the unique identifier for the formula being played.
- "vs_pc" is a boolean indicating whether the Player vs Player or Player vs PC mode has been selected.

- "turn" indicates the current turn in the game.
- "F" is the representation of the Formula class corresponding to the "formula_-filename" field.
- "U", "V", and "A" are sets initialized as in the game. Note that the notation [Formula(self.F.formula_id, [self.F.copy()], bottom)] simply means $F \to \bot$. Also, note that although "A" is theoretically a set of atomic formulas, it's more functional in the game to identify it as a set of propositional letters.

To start playing, the frontend begins by calling the NewGame endpoint to set up the game. When calling it, it passes the formula with which it wishes to play: if valid, the backend initializes the game. The FrontEnd then uses the NextTurn endpoint, passing the player's move, to proceed in the game.

```
game_status: Game = game_sessions[game_session]

if game_status.is_opponent_turn():
    logger.info("Came from opponent's turn")
    game_status.apply_opponent_actions(formula_id)
else:
    logger.info("Came from proponent's turn")
    game_status.apply_proponent_actions(formula_id)
game_status.next_turn()
```

In the provided code, the various checks that the program performs to prevent incorrect definitions are not included.

Within the Game class, the following methods are defined:

```
def apply_opponent_actions(self, formula_id: int):
    logger.debug(f"formula id: {formula_id}")
    logger.debug(f"set: {self.V}")
    selected_formula = self.V[formula_id]
    self.V.remove(selected_formula)
    self.V = []

    for hypo in selected_formula.hypo:
        if hypo not in self.U:
            self.U.append(deepcopy(hypo))
    if selected_formula.conclusion not in self.A:
        self.A.append(deepcopy(selected_formula.conclusion))

def apply_proponent_actions(self, formula_id: int):
    logger.debug(f"formula id: {formula_id}")
    logger.debug(f"set: {self.U}")
    selected_formula = self.U[formula_id]
    self.V = selected_formula.hypo.copy()
```

The representation that the frontend provides to the backend of the turn, the FrontEnd knows what to display through the Json.

```
def get_turn_context(self) -> dict:

    context = {
        'winner': False,
        'turn': self.turn,
        'player': self.player_turn,
        'formula': self.F.context,
        'set': 'V' if self.is_opponent_turn() else 'U',
        'U': [{"id": i, **f.context, "valid": f.conclusion in self.A} for i, f
        'V': [{"id": i, **f.context, "valid": True} for i, f in enumerate(self
        'A': [f.html for f in self.A]
    }

    if self.is_opponent_turn() and len(self.V) == 0:
        context['winner'] = True

    return context
```

'formula.context' calls this property

```
@property
def context(self):
    return {
        "hypothesis": [h.html for h in self.hypo],
        "placeholder": PlaceholderEnum.RIGHT_ARROW.placeholder,
        "conclusion": self.conclusion.html,
        "name": self.formula_name
    }
```

A.2. First Order Logic

First Order Logic mode clearly shares many characteristics with Propositional Logic mode; however, the introduction of variables and theories will change the game dynamics.

A.2.1. Definition of the Formula Class

The first step is to define, through Json, a hierarchical and recursive structure as the representation of the formula. Taking up the normal form of Krivine, a formula $\forall(\vec{x})(F_1,\ldots,F_n \to F_0)$ is a dictionary that has four main keys: "variables" are the first-level variables, `predicate_letter`, `num_of_hypos`, `hypothesis`. `predicate_letter` is the conclusion F_0 (which can be $\perp$ or a letter), `num_of_hypos` is n, *i.e.*, the number of hypotheses, "hypothesis" is in turn a dictionary that has n keys, where the value associated with the key i is the dictionary representation of F_i.

As an example, here is the dictionary representation of the formula: $\forall x(\forall y R(y) \rightarrow R(x))$

```
{
    "formula": {
        "num_variables": 1,
        "variables": [
            "x"
        ],
        "predicate_letter": "R",
        "pl_arity": 1,
        "pl_vars": [
            "x"
        ],
        "num_of_hypos": 1,
        "hypothesis": {
            "1": {
                "num_variables": 1,
                "variables": [
                    "y"
                ],
                "predicate_letter": "R",
                "pl_arity": 1,
                "pl_variables": [
                    "y"
                ],
                "num_of_hypos": 0,
                "hypothesis": {}
            }
        }
    },
    "total variables": [
        "x",
        "y"
    ],
    "creator": "Luigi",
    "formula_name": "6",
    "theory": "void"
}
```

It should be noted that in the Json of the main formula, the keys "creator" and "formula_name" are also present solely for display purposes, and "theory" and "total variables" because they define overall characteristics of the entire formula.

The Json is translated into a data structure to make it manipulable within the code. Specifically, the Formula class is defined as follows:

```
class Formula:
    def __init__(self, formula_id, variables: List[Variable], hypo: List['Formula']
                 function_status: FunctionStatus, **kwargs):
        self.formula_id = formula_id
        self.variables: List[Variable] = variables
        self.hypo: List['Formula'] = hypo
        self.conclusion: PredicateLetter = conclusion
        self.function_status: FunctionStatus = function_status

        self.creator = kwargs.get("creator", None)
        self.formula_name = kwargs.get("formula_name", None)
        self.theory = kwargs.get("theory", None)
```

The parameters that identify a formula and thus serve to generate the associated object are:

- "Formula_id" is a unique identifier for the formula
- "variables" is a list of Variable class objects of the first level quantifier
- "hypo" is a list of Formula class objects, with which recursion is generated
- "conclusion" which is a PredicateLetter
- "creator" is a string, empty by default
- "formula_name" is a string, empty by default
- "theory" is the theory where the formula is written

The Variable class is a subclass of the FuncComponent class (terms), which is an abstract class that defines similar computational behaviors among variables, constants, and, in the presence of a theory, all symbols for functions in general.

```
class Variable(FuncComponent):

    def __init__(self, index: int):
        self._type = LetterType.VARIABLE
        self._letters = VARIABLE_LETTERS
        self._letter = "x"
        self.index = index
```

In ‘type‘, it is identified that it is a FuncComponent of type ‘VARIABLE‘. ‘Index‘ is a natural number that identifies the variable. ‘Letters‘ and ‘letter‘ are only useful for methods that will later display the variables on the screen: the first variables are represented as x, y, z, ... and then continue to be represented as x_{index}.

In particular:

```
    def __str__(self):
        if self.index < len(self._letters):
            return self._letters[self.index]
        return f"{self._letter}_{self.index}"

class Constant(FuncComponent):

    def __init__(self, value: int, theory: str):
        self._type = LetterType.CONSTANT
        self.value = value

        if theory == "pa":
            self.repr = str(value)
        else:
            _letter = "c"
            if self.value < len(CONSTANT_LETTERS):
                self.repr = CONSTANT_LETTERS[self.value]
            else:
                self.repr = f"{_letter}_{self.value}"

class Function(FuncComponent):
    def __init__(self, symbol: str, arity: int, args: List[FuncComponent], theory:
                 inner: bool = False):
        self._type = LetterType.FUNCTION
        self.arity = arity
        self.symbol = symbol
        self.args = args
        self.inner = inner
        self.theory = theory
        self.func_shortcut_name = func_shortcut_name
```

PredicateLetters are a class that allows for the definition of atomic formulas that serve as the conclusion of a certain formula.

```
class PredicateLetter:
    def __init__(self, symbol: str, arity: int, arguments: List[FuncComponent], se
        self.symbol = symbol
        self.arity = arity
        self.arguments = arguments
```

```
        self.settings = settings.get(symbol, {})
        self.rep = self.settings.get("symbol", symbol)
        self.special_repr = self.settings.get("special_repr", False)

        assert len(self.arguments) == self.arity
```

A PredicateLetter is identified by a symbol, an arity, and a list of arguments equal to the arity. Indeed, these arguments can be FuncComponents, that is, closed terms. Settings are configurations that only affect the graphical aspect of the PredicateLetter: for example, $x = y$ is represented as such and not as $=(x, y)$.

A.2.2. The Rules of the Game

The rules of the game are thus defined. The game is a class that keeps track of the ongoing match.

```
class Game:
    def __init__(self, session: str, formula_filename: str, theory: str,
                 vs_pc: bool = True,
                 use_shortcuts: bool = False):
        self.session = session
        self.vs_pc = vs_pc
        self.use_shortcuts: bool = use_shortcuts
        self.theory = theory
        from first_order_logic.theory import get_theory
        self.theory_settings = get_theory(theory).get_settings()

        self.turn = 0
        formula_folder = os.path.join(ASSETS_DIR, GAME_NAME, "formulas", theory)
        self.formula_path = os.path.join(formula_folder, formula_filename + ".Json"
        formula_Json = Json.load(open(self.formula_path))

        self.F: Formula = Formula.from_Json(formula_filename, formula_Json['formula
                                            first_level=True,
                                            gen_variables=formula_Json['variables']
                                            creator=formula_Json.get("creator", "")
                                            formula_name=formula_Json.get("name", "
                                            theory=formula_Json.get("theory", theor
        self.function_status: FunctionStatus = self.F.function_status
        self.variables_constants = []
        self.new_constants = []

        self.U: List[Formula] = [Formula(self.F.formula_id, [], [self.F.copy()], bo
        self.V: List[Formula] = [self.F.copy()]
```

```
self.A: List[PredicateLetter] = [bottom]

t_folder = os.path.join(ASSETS_DIR, GAME_NAME, "formulas", theory, "T")
self.T: List[Formula] = []
if os.path.exists(t_folder):
    for t in os.listdir(t_folder):
        t_Json = Json.load(open(os.path.join(t_folder, t)))

        # remove axioms if shortcut
        t_formula_name = t_Json.get("formula_name")
        if self.use_shortcuts and t_formula_name in self.theory_settings.ge
            logger.debug(f"Skipping {t_formula_name} because it is not vali
            continue

        t_formula = Formula.from_Json(t, t_Json['formula'],
                                      first_level=True,
                                      gen_variables=t_Json['variables'],
                                      creator=t_Json.get("creator", ""),
                                      formula_name=t_Json.get("name", ""),
                                      theory=t_Json.get("theory", theory))
        self.T.append(t_formula)

self.shortcuts = {
    "equal": True
}
from first_order_logic.theory import get_theory
self.settings = get_theory(self.theory).get_settings()
```